Common Core State Standards Initiative

COURSE DESCRIPTION

Algebra 2 Semester 1

Will meet graduation requirements for Algebra 2

Subject Area: Mathematics

Course Number: 1200330

Course Title: Algebra 2 Semester 1

Credit: 0.5
American Worldwide Academy’s math course, *AWA Algebra 2*, focuses on the fundamental skills that are necessary for understanding the basics of algebra. This Study guide addresses essential standards of polynomials, rational expressions and radical expressions. *AWA Algebra 2* is full of practical, useful information geared to helping students recover credit for algebra while mastering the basics. This Study guide will be helpful to any student who has previously had difficulties with understanding algebraic concepts and skills.

There are six sections that cover core topics of algebra 2 at the course level. At the beginning of each section of study, you will see the objectives outlined that will help you master the standards for the section.
Course Objectives
After completion of this course, students will know and be able to do the following:

Algebra Standards and Concepts

Section 1: Polynomials -
Students perform operations on polynomials. They find factors of polynomials, learning special
techniques for factoring quadratics. They understand the relationships among the solutions of
polynomial equations, the zeros of a polynomial function, the x-intercepts of a graph, and the factors
of a polynomial.

- Simplify monomials and monomial expressions using the laws of integral exponents.
- Add, subtract, and multiply polynomials.
- Factor polynomial expressions.
- Divide polynomials by monomials and polynomials with various techniques, including synthetic
division.
- Graph polynomial functions with and without technology and describe end behavior.
- Use theorems of polynomial behavior (including but not limited to the Fundamental Theorem
of Algebra, Remainder Theorem, the Rational Root Theorem, Descartes.
- Write a polynomial equation for a given set of real and/or complex roots.
- Describe the relationships among the solutions of an equation, the zeros of a function, the
x-intercepts of a graph, and the factors of a polynomial expression, with and without
technology.
- Use polynomial equations to solve real-world problems.
- Solve a polynomial inequality by examining the graph with and without the use of technology.
- Apply the Binomial Theorem

Section 2: Rational Expressions and Equations –
Students simplify rational expressions and solve rational equations using what they have learned
about factoring polynomials.

- Simplify algebraic ratios.
- Add, subtract, multiply, and divide rational expressions.
- Simplify complex fractions.
- Solve algebraic proportions.
- Solve rational equations.
- Identify removable and non-removable discontinuities and vertical, horizontal, and oblique
asymptotes of a graph of a rational function, find the zeros, and graph the function.

Section 2: Quadratic Equations –
Students simplify and perform operations on radical expressions and equations. They also rationalize
square root expressions and understand and use the concepts of negative and rational exponents.
They add, subtract, multiply, divide, and simplify radical expressions and expressions with rational
exponents. Students will solve radical equations and equations with terms that have rational
exponents.

- Simplify radical expressions.
- Add, subtract, multiply and divide radical expressions (square roots and higher).
- Simplify expressions using properties of rational exponents.
- Convert between rational exponent and radical forms of expressions.
- Solve equations that contain radical expressions.
Getting Started

You will learn much from this course that will help you in your future studies and career. In addition to reviewing and completing the study guide and textbook, your Final Examination will be evidence that you have mastered the standards for algebra. You will know the concepts and be able to do the skills that will earn you one half credit for Algebra 2.

If you are ready to begin, turn to the next page in this Study guide: the Progress Chart and Self-Test Schedule, which will serve as a guide to help you move through the course. Let’s get started on earning that algebra credit—good luck!
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Content Review for Section 1: Polynomials

❖ Monomials

A monomial is an algebraic expression that consists of only one term. (A term is a numerical or literal expression with its own sign.) For instance, \(9x, 4a^2\), and \(3mpx^2\) are all monomials. The number in front of the variable is called the numerical coefficient. In \(9x\), 9 is the coefficient.

Adding and subtracting monomials

To add or subtract monomials, follow the same rules as with signed numbers, provided that the terms are alike. Notice that you add or subtract the coefficients only and leave the variables the same.

Example 1

Perform the operation indicated.

\[
\begin{align*}
1. & \quad 15x^2yz - 18x^2yz = -3x^2yz \\
2. & \quad 3x + 2x = 5x \\
3. & \quad \frac{93}{-3} = -31 \\
4. & \quad 17q + 8q - 3q - (-4q) = 22q - (-4q) = 22q + 4q = 26q
\end{align*}
\]

Remember that the rules for signed numbers apply to monomials as well.

Multiplying monomials

Reminder: The rules and definitions for powers and exponents also apply in algebra.

\[
5 \cdot 5 = 5^1 \quad \text{and} \quad x \cdot x = x^1
\]

Similarly, \(a \cdot a \cdot a \cdot b \cdot b = a^3b^2\).
To multiply monomials, add the exponents of the same bases.

Example 2

Multiply the following.

1. \((x^3)(x^4) = x^{3+4} = x^7\)
2. \((x^2y)(x^3y^2) = (x^2x^3)(yy^2) = x^{2+3}y^{1+2} = x^5y^3\)
3. \((6k^5)(5k^2) = (6 \times 5)(k^5k^2) = 30k^{5+2} = 30k^7\) (multiply numbers)
4. \((-4(m^2n)(-3m^4n^3) = [(-4)(-3)](m^2m^4)(nn^3) = 12m^{2+4}n^{1+3} = 12m^6n^4\) (multiply numbers)
5. \((c^2)(c^3)(c^4) = c^{2+3+4} = c^9\)
6. \((3a^2b^3c)(b^2c^2d) = 3(a^2)(b^3b^2)(cc^2)(d) = 3a^2b^{3+2}c^{1+2}d = 3a^2b^5c^3d\)

Note that in example (d) the product of \(-4\) and \(-3\) is +12, the product of \(m^2\) and \(m^4\) is \(m^6\), and the product of \(n\) and \(n^3\) is \(n^4\), because any monomial having no exponent indicated is assumed to have an exponent of \(1\).

When monomials are being raised to a power, the answer is obtained by multiplying the exponents of each part of the monomial by the power to which it is being raised.

Example 3

Simplify.

1. \((a^7)^3 = a^{21}\)  
2. \((x^3y^2)^4 = x^{12}y^8\)  
3. \((2x^2y^3)^3 = (2)^3x^6y^9 = 8x^6y^9\)

Dividing monomials - To divide monomials, subtract the exponent of the divisor from the exponent of the dividend of the same base.

Example 4-Divide.

1. \(\frac{y^{15}}{y^4} = y^{11}\) or \(y^{15} \div y^4 = y^{11}\)
2. \(\frac{x^5y^2}{x^3y} = x^2y\)
3. \(\frac{36a^4b^6}{-9ab} = -4a^3b^5\) (divide the numbers)
4. \(\frac{fg^{15}}{g^3} = fg^{12}\)
5. \(\frac{x^5}{x^8} = \frac{1}{x^3}\) (may also be expressed as \(x^{-3}\))
6. \(-\frac{3(xy)(xy^2)}{xy}\)

You can simplify the numerator first.

\[-\frac{3(xy)(xy^2)}{xy} = -3x^2y^3\]

Or, because is the numerator is all multiplication, you can reduce,

\[-\frac{3\left(\frac{xy}{xy}\right)(xy^2)}{xy} = -3xy^2\]

**Working with negative exponents**

Remember, if the exponent is negative, such as \(x^{-3}\), then the variable and exponent may be dropped under the number 1 in a fraction to remove the negative sign as follows.

\[x^{-3} = \frac{1}{x^3}\]

**Example 5**

Express the answers with positive exponents.

1. \(a^{-3}b = \frac{b}{a^3}\)

2. \(\frac{a^{-3}}{b^4} = \frac{1}{a^3b^4}\)

3. \(\left(a^2b^{-3}\right)\left(a^{-1}b^4\right) = ab\)

\[\left(\frac{a^2}{a} - a^{-1}\right) = a\]

\[\frac{b^{-3}}{b^4} = b\]
Polynomials

A polynomial consists of two or more terms. For example, \( x + y, y^2 - x^2, \) and \( x^2 + 3x + 5y^2 \) are all polynomials. A binomial is a polynomial that consists of exactly two terms. For example, \( x + y \) is a binomial. A trinomial is a polynomial that consists of exactly three terms. For example, \( y^2 + 9y + 8 \) is a trinomial.

Polynomials usually are arranged in one of two ways. Ascending order is basically when the power of a term increases for each succeeding term. For example, \( x + x^2 + x^3 \) or \( 5x + 2x^2 - 3x^3 + x^5 \) are arranged in ascending order. Descending order is basically when the power of a term decreases for each succeeding term. For example, \( x^3 + x^2 + x \) or \( 2x^4 + 3x^3 + 7x \) are arranged in descending order. Descending order is more commonly used.

Adding and subtracting polynomials

To add or subtract polynomials, just arrange like terms in columns and then add or subtract. (Or simply add or subtract like terms when rearrangement is not necessary.)

Example 1 - Do the indicated arithmetic.

Add the polynomials.

\[
\begin{align*}
1. \quad & \frac{a^2 + ab + b^2}{3a^2 + 4ab - 2b^2} \quad (5y - 3x) + (9y + 4x) = \\
& \frac{4a^2 + 5ab - b^2}{(5y - 3x) + (9y + 4x) = (5y + 9y) + (-3x + 4x) = 14y + x \text{ or } x + 14y}
\end{align*}
\]

Subtract the polynomials.

\[
\begin{align*}
3. \quad & \frac{x^2 + b^2}{-(2a^2 - b^2)} \rightarrow \frac{-2a^2 - b^2}{-a^2 + 2b^2} \quad (3cd - 6mt) - (2cd - 4mt) = \\
& \frac{(3cd - 6mt) + (-2cd - 4mt) = (3cd - 2cd) + (-6mt + 4mt) =}
\end{align*}
\]

\[
\begin{align*}
4. \quad & \frac{cd - 2mt}{3a^2bc + 2ab^2c + 4a^2bc + 5ab^2c} = \\
& \frac{3a^2bc + 2ab^2c + 4a^2bc + 5ab^2c}{7a^2bc + 7ab^2c} = \frac{7a^2bc + 7ab^2c}{7a^2bc + 7ab^2c}
\end{align*}
\]
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Multiplying polynomials

To multiply polynomials, multiply each term in one polynomial by each term in the other polynomial. Then simplify if necessary.

Example 2 - Multiply.

\[
\begin{array}{c}
2x - 2a \\
\times \\
3x + a \\
\end{array}
\]

2

\[
\begin{array}{c}
+2ax - 2a^2 \\
similar to \\
6x^2 - 6ax \\
6x^2 - 4ax - 2a^2 \\
\end{array}
\]

Or you may want to use the “F.O.I.L.” method with binomials. F.O.I.L. means First terms, Outside terms, Inside terms, Last terms. Then simplify if necessary.

Example 3

Multiply \((3x + a)(2x - 2a)\)

Multiply first terms from each quantity.

\[
\begin{array}{c}
\downarrow \\
\text{first} \\
\end{array}
\]

\[
(3x + a)(2x - 2a) = 6x^2 \\
\]

Now outside terms.

\[
\begin{array}{c}
\downarrow \\
\text{outside} \\
\end{array}
\]

\[
(3x + a)(2x - 2a) = 6x^2 - 6ax \\
\]

Now inside terms.

\[
\begin{array}{c}
\downarrow \\
\text{inside} \\
\end{array}
\]

\[
(3x + a)(2x - 2a) = 6x^2 - 6ax + 2ax \\
\]

Finally last terms.

\[
\begin{array}{c}
\downarrow \\
\text{last} \\
\end{array}
\]

\[
(3x + a)(2x - 2a) = 6x^2 - 6ax + 2ax - 2a^2 \\
\]

Now simplify.

\[
6x^2 - 6ax + 2ax - 2a^2 = 6x^2 - 4ax - 2a^2
\]
Example 4 Multiply.

\[(x + y)(x + y + z) = (x + y) \cdot x + (x + y) \cdot y + (x + y) \cdot z\]

\[= x^2 + xy + xz + xy + y^2 + yz + xz + yz + z^2\]

\[= x^2 + 2xy + 2xz + y^2 + 2yz\]

Dividing polynomials by monomials

To divide a polynomial by a monomial, just divide each term in the polynomial by the monomial.

Example 5 - Divide.

1. \[(6x^2 + 2x) \div (2x) = \frac{6x^2 + 2x}{2x} = 3x + 1\]

2. \[(16a^7 - 12a^3) \div (4a^3) = \frac{16a^7}{4a^3} - \frac{12a^3}{4a^3} = 4a^4 - 3a^3\]

Dividing polynomials by polynomials

To divide a polynomial by a polynomial, make sure both are in descending order; then use long division. (Remember: Divide by the first term, multiply, subtract, bring down.)

Example 6 - Divide \(4a^2 + 18a + 8\) by \(a + 4\).

First divide \(a\) into \(4a^2\)

\[\frac{4a}{a + 4} \]

Now multiply \(4a\) times \((a + 4)\)

\[\frac{4a}{a + 4} \frac{4a^2 + 18a + 8}{4a^2 + 16a}\]

Now subtract.

\[\frac{4a}{a + 4} \frac{4a^2 + 18a + 8}{2a}\]
Example 7 - Divide.

1. \((3x^2 + 4x + 1) + (x + 1)\)

\[
\begin{align*}
3x + 1 & \quad \text{[\(3x^2 + 4x + 1\)]} \\
- \{3x^2 + 3x\} & \quad \text{[\(-3x^2 - 3x\)]} \\
\hline
x & \quad \text{[\(x\)]} \\
\hline
0 & \quad \text{[\(0\)]}
\end{align*}
\]

2. \((2x + 1 + x^3) ÷ (x + 1) =

First change to descending order:
\(x^2 + 2x + 1\). Then divide.
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3. \( (m^3 - m) + (m + 1) = \)

\[
\frac{m^3 - m}{m + 1} \div \frac{m^3 + 0m^2 - m}{-(m^2 + m^2)} = \frac{-m^3 - m}{-(m^2 - m) + 0}
\]

Note: When terms are missing, be sure to leave proper room between terms.

4. \( [10a^2 - 29a - 21] + (2a - 7) = \)

\[
\frac{5a + 3}{2a - 7} \div \frac{10a^2 - 29a - 21}{-(10a^2 - 35a)} = \frac{6a - 21}{-(6a - 21) + 0}
\]

5. \( (x^2 + 2x + 4) + (x - 1) = \)

\[
x + 1 \text{ (with remainder 3)} \]
\[
x + 1 \div x^2 + 2x + 4 = \frac{-x^2 - x}{x + 4} = \frac{-(x + 1)}{3}
\]

This answer can be rewritten as \( x + 1 \)

\[\text{❖ Factoring}\]

To **factor** means to find two or more quantities whose product equals the original quantity.

**Factoring out a common factor**

To factor out a common factor, (1) find the largest common monomial factor of each term and (2) divide the original polynomial by this factor to obtain the second factor. The second factor will be a polynomial.

**Example 1 - Factor.**

1. \( 5x^2 + 4x = x(5x + 4) \)
2. \( 2y^3 - 6y = 2y( y^2 - 3) \)
3. \( x^5 - 4x^3 + x^2 = x^2( x^3 - 4x + 1) \)

When the common monomial factor is the last term, 1 is used as a place holder in the second factor.
Factoring the difference between two squares

To factor the difference between two squares, (1) find the square root of the first term and the square root of the second term and (2) express your answer as the product of the sum of the quantities from Step 1 times the difference of those quantities.

Example 2 - Factor.

1. \( x^2 - 144 = (x + 12)(x - 12) \)
2. \( a^2 - b^2 = (a + b)(a - b) \)
3. \( 9y^2 - 1 = (3y + 1)(3y - 1) \)

Note: \( x^2 + 144 \) is not factorable.

Factoring polynomials having three terms of the form \( ax^2 + bx + c \)

To factor polynomials having three terms of the form \( ax^2 + bx + c \), (1) check to see whether you can monomial factor (factor out common terms). Then if \( a = 1 \) (that is, the first term is simply \( x^2 \)), use double parentheses and factor the first term. Place these factors in the left sides of the parentheses. For example, \((x)(x)\)

(2) Factor the last term and place the factors in the right sides of the parentheses.

To decide on the signs of the numbers, do the following. If the sign of the last term is negative, (1) find two numbers (one will be a positive number and the other a negative number) whose product is the last term and whose difference is the coefficient (number in front) of the middle term and (2) give the larger of these two numbers the sign of the middle term and the opposite sign to the other factor.

If the sign of the last term is positive, (1) find two numbers (both will be positive or both will be negative) whose product is the last term and whose sum is the coefficient of the middle term and (2) give both factors the sign of the middle term.

Example 3 - Factor \( x^2 - 3x - 10 \).

First check to see whether you can monomial factor (factor out common terms). Because this is not possible, use double parentheses and factor the first term as follows: \((x)(x)\). Next, factor the last term, 10, into 2 times 5 (5 must take the negative sign and 2 must take the positive sign because they will then total the coefficient of the middle term, which is \(-3\)) and add the proper signs, leaving \((x - 5)(x + 2)\)

Multiply means (inner terms) and extremes (outer terms) to check.

\[
\begin{array}{c}
\text{inner} \\
\uparrow \\
(x - 5)(x + 2) \\
\downarrow \\
\text{outer}
\end{array}
\]

\(2x - 5x = -3x\) (which is the middle term).

To completely check, multiply the factors together.

\[
\begin{array}{c}
\text{inner} \\
\uparrow \\
(x - 5) \\
\downarrow \\
\text{extremes} \\
\times \times & 2 \times 1 \\
\text{middle} & +2x - 10 \\
\text{factor} & \frac{+x^2 - 5x}{x^2 - 3x - 10}
\end{array}
\]
Example 4 - Factor $x^2 + 8x + 15$.

$$(x + 3)(x + 5)$$

Notice that $3 \times 5 = 15$ and $3 + 5 = 8$, the coefficient of the middle term. Also note that the signs of both factors are $+$, the sign of the middle term. To check,

\[
\begin{array}{c}
\text{outer} \\
\downarrow \\
\downarrow \\
(x + 3)(x + 5) \\
\uparrow \\
\uparrow \\
\text{inner} \\
\end{array}
\]

$5x + 3x = 8x$ (which is the middle term)

Example 5 - Factor $x^2 - 5x - 14$.

$$(x - 7)(x + 2)$$

Notice that $7 \times 2 = 14$ and $7 - 2 = 5$, the coefficient of the middle term. Also note that the sign of the larger factor, 7, is $-$, while the other factor, 2, has a $+$ sign. To check,

\[
\begin{array}{c}
\text{outer} \\
\downarrow \\
\downarrow \\
(x - 7)(x + 2) \\
\uparrow \\
\uparrow \\
\text{inner} \\
\end{array}
\]

$2x - 7x = -5x$ (which is the middle term)

If, however, $a \neq 1$ (that is, the first term has a coefficient—for example, $4x^2 + 5x + 1$), then additional trial and error will be necessary.

Example 6 - Factor $4x^2 + 5x + 1$.

$(2x + \_)(2x + \_)$ might work for the first term. But when 1s are used as factors to get the last term, $(2x + 1)(2x + 1)$, the middle term comes out as $4x$ instead of $5x$.

\[
\begin{array}{c}
\text{outer} \\
\downarrow \\
\downarrow \\
(2x + 1)(2x + 1) \\
\uparrow \\
\uparrow \\
\text{inner} \\
\end{array}
\]

$2x + 2x = 4x$

Try $(4x + \_)(x + \_)$. Now using 1s as factors to get the last terms gives $(4x + 1)(x + 1)$. Checking for the middle term,

\[
\begin{array}{c}
\text{outer} \\
\downarrow \\
\downarrow \\
(4x + 1)(x + 1) \\
\uparrow \\
\uparrow \\
\text{inner} \\
\end{array}
\]

Therefore, $4x^2 + 5x + 1 = (4x + 1)(x + 1)$. 
Example 7 - Factor $4a^2 + 6a + 2$.

Factoring out a 2 leaves; $2(2a^2 + 3a + 1)$

Now factor as usual, giving; $2(2a + 1)(a + 1)$

To check,

Example 8 - Factor $5x^3 + 6x^2 + x$.

Factoring out an $x$ leaves: $x(5x^2 + 6x + 1)$

Now factor as usual, giving: $x(5x + 1)(x + 1)$

To check,

Example 9 - Factor $5 + 7b + 2b^2$ (a slight twist).

$(5 + 2b)(1 + b)$

To check,

Note that $(5 + b)(1 + 2b)$ is incorrect because it gives the wrong middle term.

Example 10 - Factor $x^2 + 2xy + y^2$.

$(x + y)(x + y)$

To check,
Example 11 - Factor $3x^2 - 48$.

Factoring out a 3 leaves: $3(x^2 - 16)$

But $x^2 - 16$ is the difference between two squares and can be factored into $(x + 4) (x - 4)$. Therefore, when completely factored, $3x^2 - 48 = 3(x + 4) (x - 4)$.

Factoring by grouping

Some polynomials have binomial, trinomial, and other polynomial factors.

Example 12 - Factor $x + 2 + xy + 2y$.

Since there is no monomial factor, you should attempt rearranging the terms and looking for binomial factors.

Grouping gives; $(x + xy) + (2 + 2y)$

Now factoring gives; $x(1 + y) + 2(1 + y)$

Using the distributive property gives $(x + 2)(1 + y)$

You could rearrange them differently, but you would still come up with the same factoring.

Summary of the factoring methods

When factoring polynomials, you should look for factoring in the following order.

1. Look for the greatest common factor if one exists.
2. If there are two terms, look for the difference of square numbers.
3. If there are three terms, look for a pattern that applies to $ax^2 + bx + c$.
4. If there are four or more terms, look for some type of regrouping that will lead to other factoring.

Note: There are polynomials that are not factorable.

Example 13 - Factor $2x^2 + 3x + 5$.

1. This polynomial does not have a common factor.
2. This polynomial is not a difference of square numbers.
3. There is no $(\_ \_ x) (\_ \_ x)$ combination that produces $2x^2 + 3x + 5$.
4. Since there are only three terms, there is no regrouping possibility.

Therefore, this polynomial is not factorable.
Graphing Polynomial Functions

Polynomial functions of the form $f(x) = x^n$ (where $n$ is a positive integer) form one of two basic graphs, shown in Figure 1.

Each graph has the origin as its only $x$-intercept and $y$-intercept. Each graph contains the ordered pair $(1, 1)$. If a polynomial function can be factored, its $x$-intercepts can be immediately found. Then a study is made as to what happens between these intercepts, to the left of the far left intercept and to the right of the far right intercept.

**Example 1** - Graph $f(x) = x^4 - 10x^2 + 9$.

The zeros of this function are $-1, 1, -3,$ and $3$. That is, $-1, 1, -3,$ and $3$ are the $x$-intercepts of this function.

When $x < -3$, say, $x = -4$, then

$$f(-4) = (-4 + 1)(-4 - 1)(-4 + 3)(-4 - 3)$$
$$= (-3)(-5)(-1)(-7)$$
$$= 105$$

So for $x < -3, f(x) > 0$.

When $-1 < x < 1$, say, $x = 0$, then

$$f(0) = (0 + 1)(0 - 1)(0 + 3)(0 - 3)$$
$$= (1)(-1)(3)(-3)$$
$$= 9$$

So for $-1 < x < 1, f(x) > 0$. 

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In a similar way, it can be seen that

- when \( x > 3 \), \( f(x) > 0 \)
- when \( -3 < x < -1 \), \( f(x) < 0 \)
- when \( 1 < x < 3 \), \( f(x) < 0 \)

The graph then has points in the shaded regions as shown in Figure 2.

The \( y \)-intercept of this function is found by finding \( f(0) \).

\( f(0) = 9 \)

so \((0, 9)\) is a point on the graph. To complete the graph, find and plot several points. Evaluate \( f(x) \) for several integer replacements; then connect these points to form a smooth curve (see Figure 3).

Notice that \( f(x) = x^4 - 10x^2 + 9 \) has a leading term with an even exponent. The far right and far left sides of the graph will go in the same direction. Because the leading coefficient is positive, the two sides will go up. If the leading coefficient were negative, the two sides would go down.
Example 2

Graph \( f(\ x) = x^3 - 19x + 30. \)

\( f(\ x) = x^3 - 19x + 30 \) can be factored using the rational zero theorem:

\[
\begin{array}{c|c|c|c|c}
 p/q & 1 & 0 & -19 & 30 \\
\hline
 1 & 1 & 1 & -18 & 12 \\
-1 & 1 & -1 & -18 & 48 \\
 2 & 1 & 2 & -15 & 0 \\
\end{array}
\]

\( f(\ x) \) can now be written in factored form and further factored.

\[
= (x - 2)(x - 3)(x + 5)
\]

The zeros of this function are 2, 3, and -5 (see Figure 4).

Notice that \( f(\ x) = x^3 - 19x + 30 \) has a leading term that has a positive coefficient and an odd exponent. This function will always go up toward the far right and down toward the far left. If the leading coefficient were negative with an odd exponent, the graph would go up toward the far left and down toward the far right.

![Figure 4. A cubic equation.](image-url)
General Theorems for Polynomials

1) **Fundamental Theorem of Algebra**
If \( P(x) \) is a polynomial function of degree \( n \) \((n > 0)\) with complex coefficients, then the equation \( P(x) = 0 \) has \( n \) roots assuming you count double roots as 2, triple roots as 3, etc.

2) **Complex Conjugates Theorem**
If \( P(x) \) is a polynomial function with real coefficients, and \( a + bi \) is a solution of the equation \( P(x) = 0 \), then \( a - bi \) is also a solution.

3) If \( P(x) \) is a polynomial with rational coefficients and \( a \) and \( b \) are also rational such that the square root of \( b \) is irrational, then if \( a + \sqrt{b} \) is a root of the equation \( P(x) = 0 \), then \( a - \sqrt{b} \) is also a root.

4) If \( P(x) \) is a polynomial of odd degree with real coefficients, then the equation \( P(x) = 0 \) has at least one real solution.

5) For a polynomial equation with \( a_n \) as the leading coefficient and \( a_0 \) as the constant then the following is true:
   a) the sum of the roots is \(-a_{n-1}/a_n\)
   b) the product of the roots is:
      \[ \frac{a_0}{a_n} \text{ if } n \text{ is even} \]
      \[ -\frac{a_0}{a_n} \text{ if } n \text{ is odd} \]

A useful formula to help you find equations given the root is:
\[ x^2 - (\text{sum of the roots})x + (\text{product of the roots}) = 0 \]

Sample Problems

1) Find a quadratic equation with the root \( 2 + 5i \)
   Solution:
   Since complex solutions come in pairs, \( 2 - 5i \) is also a solution. Find the sum and product and use the above formula.
   Sum = \((2 + 5i) + (2 - 5i) = 4\)
   Product = \((2 + 5i)(2 - 5i) = 4 - 25i^2 = 4 + 25 = 29\);
   Therefore, the equation is: \( x^2 - 4x + 29 = 0 \)

2) Find a cubic equation with integral coefficients for \( 3 + i \) and 2.
   Solution:
   Again, complex solutions come in pairs. \( 3 - i \) is a solution. Using the complex solutions, find a quadratic.
   Sum = \((3 + i) + (3 - i) = 6\)
   Product = \((3 + i)(3 - i) = 9 - i^2 = 9 + 1 = 10\);
   Therefore, the quadratic is \( x^2 - 6x + 10 \)
   So the equation is: \((x - 2)(x^2 - 6x + 10) = 0\)
   \( x^3 - 6x^2 + 10x - 2x^2 + 12x - 20 = 0 \)
   \( x^3 - 8x^2 + 22x - 20 = 0 \)
   Notice that the sum of the three roots is 8 and the product is 20!
3) Find a quadratic equation with the following roots: i and 2 + i

Solution:

Again, complex come in pairs. So, -i and 2 - i are also roots

Form two quadratics for the solutions and multiply them!

Quadratic #1 sum = i + -i = 0
Product = i(-i) = -i^2 = 1
First quadratic is: x^2 + 1

Quadratic #2 sum = (2 + i) + (2 - i) = 4
Product = (2 + i)(2 - i) = 4 - i^2 = 4 + 1 = 5
Second quadratic is: x^2 -4x + 5

Therefore, the equation is: (x^2 + 1)(x^2 - 4x + 5) = 0
x^4 - 4x^3 + 5x^2 + x^2 - 4x + 5 = 0
x^4 - 4x^3 + 6x^2 - 4x + 5 = 0

Graphing Polynomials

In this section we are going to look at a method for getting a rough sketch of a general polynomial. The only real information that we’re going to need is a complete list of all the zeroes (including multiplicity) for the polynomial.

In this section we are going to either be given the list of zeroes or they will be easy to find. In the next section we will go into a method for determining a large portion of the list for most polynomials. We are graphing first since the method for finding all the zeroes of a polynomial can be a little long and we don’t want to obscure the details of this section in the mess of finding the zeroes of the polynomial.

Let’s start off with the graph of couple of polynomials.

![Graphs of Polynomials](image)

Do not worry about the equations for these polynomials. We are giving these only so we can use them to illustrate some ideas about polynomials.

First, notice that the graphs are nice and smooth. There are no holes or breaks in the graph and there are no sharp corners in the graph. The graphs of polynomials will always be nice smooth curves.
Secondly, the “humps” where the graph changes direction from increasing to decreasing or decreasing to increasing are often called **turning points**. If we know that the polynomial has degree \( n \) then we will know that there will be at most \( n - 1 \) turning points in the graph. While this won’t help much with the actual graphing process it will be a nice check. If we have a fourth degree polynomial with 5 turning point then we will know that we’ve done something wrong since a fourth degree polynomial will have no more than 3 turning points.

Next, we need to explore the relationship between the \( x \)-intercepts of a graph of a polynomial and the zeroes of the polynomial. Recall that to find the \( x \)-intercepts of a function we need to solve the equation

\[
P(x) = 0
\]

Also, recall that \( x = r \) is a zero of the polynomial, \( P(r) \), provided \( P(r) = 0 \). But this means that \( x = r \) is also a solution to \( P(r) = 0 \).

In other words, the zeroes of a polynomial are also the \( x \)-intercepts of the graph. Also, recall that \( x \)-intercepts can either cross the \( x \)-axis or they can just touch the \( x \)-axis without actually crossing the axis.

Notice as well from the graphs above that the \( x \)-intercepts can either flatten out as they cross the \( x \)-axis or they can go through the \( x \)-axis at an angle.

The following fact will relate all of these ideas to the multiplicity of the zero.

**Fact**

If \( x = r \) is a zero of the polynomial \( P(x) \) with multiplicity \( k \) then,

1. If \( k \) is odd then the \( x \)-intercept corresponding to \( x = r \) will cross the \( x \)-axis.
2. If \( k \) is even then the \( x \)-intercept corresponding to \( x = r \) will only touch the \( x \)-axis and not actually cross it.

Furthermore, if \( k > 1 \) then the graph will flatten out at \( x = r \).

Finally, notice that as we let \( x \) get large in both the positive or negative sense (i.e. at either end of the graph) then the graph will either increase without bound or decrease without bound. This will always happen with every polynomial and we can use the following test to determine just what will happen at the endpoints of the graph.

**Leading Coefficient Test**

Suppose that \( P(x) \) is a polynomial with degree \( n \). So we know that the polynomial must look like,

\[
P(x) = ax^n + \cdots
\]

We don’t know if there are any other terms in the polynomial, but we do know that the first term will have to be the one listed since it has degree \( n \). We now have the following facts about the graph of \( P(x) \) at the ends of the graph.

1. If \( a > 0 \) and \( n \) is even then the graph of \( P(x) \) will increase without bound positively at both endpoints. A good example of this is the graph of \( x^2 \).
2. If \( a > 0 \) and \( n \) is odd then the graph of \( P(x) \) will increase without bound positively at the right end and decrease without bound at the left end. A good example of this is the graph of \( x^3 \).

3. If \( a > 0 \) and \( n \) is even then the graph of \( P(x) \) will decrease without bound positively at both endpoints. A good example of this is the graph of \(-x^2\).

4. If \( a > 0 \) and \( n \) is odd then the graph of \( P(x) \) will decrease without bound positively at the right end and increase without bound at the left end. A good example of this is the graph of \(-x^3\).

Okay, now that we’ve got all that out of the way we can finally give a process for getting a rough sketch of the graph of a polynomial.

**Process for Graphing a Polynomial**

1. Determine all the zeroes of the polynomial and their multiplicity. Use the fact above to determine the \( x \)-intercept that corresponds to each zero will cross the \( x \)-axis or just touch it and if the \( x \)-intercept will flatten out or not.

2. Determine the \( y \)-intercept, \((0, P(0))\).

3. Use the leading coefficient test to determine the behavior of the polynomial at the end of the graph.

4. Plot a few more points. This is left intentionally vague. The more points that you plot the better the sketch. At the least you should plot at least one at either end of the graph and at least one point between each pair of zeroes.

We should give a quick warning about this process before we actually try to use it. This process assumes that all the zeroes are real numbers. If there are any complex zeroes then this process may miss some pretty important features of the graph.

Let’s sketch a couple of polynomials.
Example 1 Sketch the graph of $P(x) = 5x^5 - 20x^4 - 5x^3 + 50x^2 - 20x - 40$.

Solution
We found the zeroes and multiplicities of this polynomial in the previous section so we'll just write them back down here for reference purposes.

\[
\begin{align*}
  x &= -1 \quad \text{(multiplicity 2)} \\
  x &= 2 \quad \text{(multiplicity 3)}
\end{align*}
\]

So, from the fact we know that $x = -1$ will just touch the x-axis and not actually cross it and that $x = 2$ will cross the x-axis and will be flat as it does this since the multiplicity is greater than 1.

Next, the y-intercept is $(0, -40)$.

The coefficient of the 5th degree term is positive and since the degree is odd we know that this polynomial will increase without bound at the right end and decrease without bound at the left end.

Finally, we just need to evaluate the polynomial at a couple of points. The points that we pick aren’t really all that important. We just want to pick points according to the guidelines in the process outlined above and points that will be fairly easy to evaluate. Here are some points. We will leave it to you to verify the evaluations.

\[
\begin{align*}
P(-2) &= -320 \\
P(1) &= -20 \\
P(3) &= 80
\end{align*}
\]

Now, to actually sketch the graph we’ll start on the left end and work our way across to the right end. First, we know that on the left end the graph decreases without bound as we make $x$ more and more negative and this agrees with the point that we evaluated at $x = -2$.

So, as we move to the right the function will actually be increasing at $x = -2$ and we will continue to increase until we hit the first x-intercept at $x = 0$. At this point we know that the graph just touches the x-axis without actually crossing it. This means that at $x = 0$ the graph must be a turning point.

The graph is now decreasing as we move to the right. Again, this agrees with the next point that we’ll run across, the y-intercept.

Now, according to the next point that we’ve got, $x = 1$, the graph must have another turning point somewhere between $x = 0$ and $x = 1$ since the graph is higher at $x = 1$ than at $x = 0$. Just where this turning point will occur is very difficult to determine at this level so we won’t worry about trying to find it. In fact, determining this point usually requires some Calculus.

So, we are moving to the right and the function is increasing. The next point that we hit is the x-intercept at $x = 2$ and this one crosses the x-axis so we know that there won’t be a turning point here as there was at the first x-intercept. Therefore, the graph will continue to increase through this point until we hit the final point that we evaluated the function at, $x = 3$.

At this point we’ve hit all the x-intercepts and we know that the graph will increase without bound at the right end and so it looks like all we need to do is sketch in an increasing curve.

Here is a sketch of the polynomial.

Okay, let’s take a look at another polynomial. This time we’ll go all the way through the process of finding the zeroes.
Example 2 Sketch the graph of \( P(x) = x^4 - x^3 + 6x^2 \).

Solution
First, we’ll need to factor this polynomial as much as possible so we can identify the zeroes and get their multiplicities.

\[ P(x) = x^4 - x^3 + 6x^2 = x^2(x - 3)(x + 2) \]

Here is a list of the zeroes and their multiplicities.

- \( x = -2 \) (multiplicity 1)
- \( x = 0 \) (multiplicity 2)
- \( x = 3 \) (multiplicity 1)

So, the zeroes at \( x = -2 \) and \( x = 3 \) will correspond to \( x \)-intercepts that cross the \( x \)-axis since their multiplicity is odd and will do so at an angle since their multiplicity is NOT at least 2. The zero at \( x = 0 \) will not cross the \( x \)-axis since its multiplicity is even. The \( y \)-intercept is \( (0,0) \) and notice that this is also an \( x \)-intercept. The coefficient of the 4th degree term is positive and so since the degree is even we know that the polynomial will increase without bound at both ends of the graph.

Finally, here are some function evaluations.

\[ P(-3) = 54 \quad P(-1) = -4 \quad P(1) = -6 \quad P(4) = 96 \]

Now, starting at the left end we know that as we make \( x \) more and more negative the function must increase without bound. That means that as we move to the right the graph will actually be decreasing. At \( x = -3 \) the graph will be decreasing and will continue to decrease when we hit the first \( x \)-intercept at \( x = -2 \) since we know that this \( x \)-intercept will cross the \( x \)-axis. Next, since the next \( x \)-intercept is at \( x = 0 \) we will have to have a turning point somewhere so that the graph can increase back up to this \( x \)-intercept. Again, we won’t worry about where this turning point actually is. Once we hit the \( x \)-intercept at \( x = 0 \) we know that we’ve got to have a turning point since this \( x \)-intercept doesn’t cross the \( x \)-axis. Therefore to the right of \( x = 0 \) the graph will now be decreasing. It will continue to decrease until it hits another turning point (at some unknown point) so that the graph can get back up to the \( x \)-axis for the next \( x \)-intercept at \( x = 3 \). This is the final \( x \)-intercept and since the graph is increasing at this point and must increase without bound at this end we are done.

Here is a sketch of the graph.

![Graph of P(x) = x^4 - x^3 + 6x^2](image-url)
Example 3 Sketch the graph of $P(x) = -x^5 + 4x^3$.

Solution
As with the previous example we’ll first need to factor this as much as possible.

$$P(x) = -x^5 + 4x^3 = -(x^5 - 4x^3) = -x^3(x - 2)(x = 2)$$

Notice that we first factored out a minus sign to make the rest of the factoring a little easier. Here is a list of all the zeroes and their multiplicities.

$$x = -2 \text{ (multiplicity 1)}$$
$$x = 0 \text{ (multiplicity 3)}$$
$$x = 2 \text{ (multiplicity 1)}$$

So, all three zeroes correspond to $x$-intercepts that actually cross the $x$-axis since all their multiplicities are odd, however, only the $x$-intercept at $x = 0$ will cross the $x$-axis flattened out.

The $y$-intercept is $(0,0)$ and as with the previous example this is also an $x$-intercept. In this case the coefficient of the 5th degree term is negative and so since the degree is odd the graph will increase without bound on the left side and decrease without bound on the right side.

Here are some function evaluations.

$$P(-3) = 135 \quad P(-1) = -3 \quad P(1) = 3 \quad P(3) = -135$$

Alright, this graph will start out much as the previous graph did. At the left end the graph will be decreasing as we move to the right and will decrease through the first $x$-intercept at $x = -2$ since know that this $x$-intercept crosses the $x$-axis.

Now at some point we’ll get a turning point so the graph can get back up to the next $x$-intercept at $x = 0$ and the graph will continue to increase through this point since it also crosses the $x$-axis. Note as well that the graph should be flat at this point as well since the multiplicity is greater than one.

Finally, the graph will reach another turning point and start decreasing so it can get back down to the final $x$-intercept at $x = 2$. Since we know that the graph will decrease without bound at this end we are done.

Here is the sketch of this polynomial.
The process that we’ve used in these examples can be a difficult process to learn. It takes time to learn how to correctly interpret the results. Also, as pointed out at various spots there are several situations that we won’t be able to deal with here. To find the majority of the turning points we would need some Calculus, which we clearly don’t have. Also, the process does require that we have all the zeroes and that they all be real numbers. Even with these drawbacks however, the process can at least give us an idea of what the graph of a polynomial will look like.

**Solving Polynomial Inequalities Analytically**

This section assumes that you know how to find the roots of a polynomial.

Let's suppose you want to solve the inequality \( x^2 - 4x + 3 < 0 \).

**Step 1.** Solve the equation \( f(x) = x^2 - 4x + 3 = 0 \).

In this case, we can "factor by guessing":

\[ x^2 - 4x + 3 = (x-1)(x-3), \]

so the roots of the equation \( f(x) = 0 \) are \( x = 1 \) and \( x = 3 \). Draw a picture of the \( x \)-axis and mark these points.

**Step 2.** Our solutions partition the \( x \)-axis into three intervals. Pick a point (your choice!) in each interval. Let me take \( x = 0 \), \( x = 2 \) and \( x = 4 \). Compute \( f(x) \) for these points:

\[
\begin{align*}
f(0) &= 0 - 0 + 3 = 3 > 0 \\
f(2) &= 4 - 8 + 3 = -1 < 0 \\
f(4) &= 16 - 16 + 3 = 3 > 0
\end{align*}
\]

These three points are representative for what happens in the intervals they are contained in:

Since \( f(0) > 0, f(x) \) will be positive for all \( x \) in the interval \((-\infty, 1)\). Similarly, since \( f(2) < 0, f(x) \) will be negative for all \( x \) in the interval \((1,3)\). Since \( f(4) > 0, f(x) \) will be positive for all \( x \) in the interval \((3, \infty)\). You can indicate this on the \( x \)-axis by inserting plus or minus signs on the \( x \)-axis. I use color coding instead: blue for positive, red for negative:
Step 3. We want to solve the inequality

\[ x^2 - 4x + 3 < 0, \]

so we are looking for all \( x \) such that \( f(x) < 0 \). Consequently, the interval \((1,3)\) contains all solutions to the inequality.

Why does this work? Let's look at the graph of \( f(x)\):

Note the pivotal role played by the "yellow dots", the \( x \)-intercepts of \( f(x)\). \( f(x) \) can only change its sign by passing through an \( x \)-intercept, \( i.e., \) a solution of \( f(x) = 0 \) will always separate parts of the graph of \( f(x) \) above the \( x \)-axis from parts below the \( x \)-axis. Thus it suffices to pick a representative in each of the three intervals separated by "yellow dots", to test whether \( f(x) \) is positive or negative in the interval.

This nice property of polynomials is called the **Intermediate Value Property** of polynomials; your teacher might also refer to this property as **continuity**.

Here is another example: Find the solutions of the inequality

\[ x^3 + 2 \geq 3x. \]

For our method to work it is essential that the right side of the inequality equals zero! So let's change our inequality to

\[ x^3 - 3x + 2 \geq 0. \]

Step 1. Solve the equation \( f(x) = x^2 - 3x + 2 = 0 \).

Again, we can "factor by guessing":

\[ x^2 - 3x + 2 = (x-1)(x-2), \]

so the roots of the equation \( f(x) = 0 \) are \( x = 1 \) and \( x = 2 \). Draw a picture of the \( x \)-axis and mark these points.
Step 2. Our solutions partition the x-axis into three intervals. Pick a point (your choice!) in each interval. Let me take x=0, x=1.5 and x=3. Compute f(x) for these points:

\[
\begin{align*}
  f(0) &= 0 - 0 + 2 > 0 \\
  f(1.5) &= 2.25 - 4.5 + 2 < 0 \\
  f(3) &= 9 - 9 + 2 > 0
\end{align*}
\]

These three points are representative for what happens in the intervals they are contained in: Since f(0)>0, f(x) will be positive for all x in the interval (−∞, 1). Similarly, since f(1.5)<0, f(x) will be negative for all x in the interval (1,2). Since f(3)>0, f(x) will be positive for all x in the interval (2, ∞). You can indicate this on the x-axis by inserting plus or minus signs on the x-axis. I use color coding instead: blue for positive, red for negative:

Step 3. We want to solve the inequality

\[x^2 - 3x + 2 \geq 0,\]

so we are looking for all x such that \( f(x) \geq 0 \). Consequently, the set \((-∞, 1] \cup [2, ∞)\) contains all solutions to the inequality. (Since our inequality only stipulates that \( f(x) \geq 0 \), x=1 and x=2 are solutions, so we include them. "∞" and "−∞" are only symbols; they will never be included as solutions.)

Our next example: Solve \(x^3 > 2x\). Do not divide by x on both sides! If you do so, you will never be able to arrive at the correct answer. Repeat the pattern instead; make one side of the inequality equal zero:

\[x^3 - 2x > 0.\]

Step 1. Solve the equation \( f(x) = x^3 - 2x = 0 \).

We can factor rather easily:

\[x^3 - 2x = x(x^2 - 2) = x(x - \sqrt{2})(x + \sqrt{2}),\]

so the roots of the equation \( f(x) = 0 \) are \( x = -\sqrt{2}, x=0 \) and \( x = \sqrt{2} \). Draw a picture of the x-axis and mark these points.
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**Step 2.** Our solutions partition the x-axis into four intervals. Pick a point (your choice!) in each interval. Let me take $x=-2, x=-1, x=1$ and $x=2$. Compute $f(x)$ for these points:

- $f(-2) = -8 + 4 < 0$
- $f(-1) = -1 + 2 > 0$
- $f(1) = 1 - 2 < 0$
- $f(2) = 8 - 4 > 0$

These four points are representative for what happens in the intervals they are contained in: Since $f(-2)<0$, $f(x)$ will be negative for all $x$ in the interval $(-\infty, -\sqrt{2})$. Similarly, since $f(-1)>0$, $f(x)$ will be positive for all $x$ in the interval $(-\sqrt{2}, 0)$. Since $f(1)<0$, $f(x)$ will be negative for all $x$ in the interval $(0, \sqrt{2})$. Since $f(2)>0$, $f(x)$ will be positive for all $x$ in the interval $(\sqrt{2}, \infty)$. You can indicate this on the x-axis by inserting plus or minus signs on the x-axis. I use color coding instead: blue for positive, red for negative:

---

**Step 3.** We want to solve the inequality

$$x^3-2x > 0,$$

so we are looking for all $x$ such that $f(x)>0$. Consequently, the set $(-\sqrt{2}, 0] \cup [\sqrt{2}, \infty)$ contains all solutions to the inequality. (Since our inequality stipulates that $f(x)>0$, $x = \pm\sqrt{2}$ do not qualify as solutions, so we exclude them. "+$\infty$" and "$-\infty$" are only symbols; they will never be included as solutions.)

**Here is my last example: Solve**

$$x^3 + 3x^2 + x + 3 \leq 0.$$  

**Step 1.** Solve the equation $f(x)=x^3+3x^2+x+3$.

We can factor either by finding a rational zero, or by clever grouping:

$$x^3+3x^2+x+3= (x^3+3x^2)+(x+3)=x^2(x+3)+(x+3)=(x^2+1)(x+3),$$

so there is only one real root of the equation $f(x)=0$, namely $x=-3$. Draw a picture of the x-axis and mark this point.

The polynomial $x^2+1$ is irreducible, it does not have real roots. Its complex roots are irrelevant for our purposes.
Step 2. Our solution partitions the x-axis into two intervals. Pick a point (your choice!) in each interval. Let me take $x=-4$ and $x=0$. Compute $f(x)$ for these points:

\[
\begin{align*}
  f(-4) &= -17 < 0 \\
  f(0) &= 3 > 0
\end{align*}
\]

These two points are representative for what happens in the intervals they are contained in: Since $f(-4)<0$, $f(x)$ will be negative for all $x$ in the interval $(-\infty, -3)$. Similarly, since $f(0)>0$, $f(x)$ will be positive for all $x$ in the interval $(-3, \infty)$. You can indicate this on the x-axis by inserting plus or minus signs on the x-axis. I use color coding instead: blue for positive, red for negative:

![Graph of positive and negative intervals](image)

Step 3. We want to solve the inequality

\[
x^3 + 3x^2 + x + 3 \leq 0.
\]

so we are looking for all $x$ such that $f(x) \leq 0$. Consequently, the set $(-\infty, -3]$ is the set of solutions to the inequality. (Since our inequality stipulates that $f(x) \leq 0$, $x=-3$ is also a solution, so we include it. "$+\infty"$ and "$-\infty" are only symbols; they will never be included as solutions.)

Exercise 1. Find the solutions of the inequality

\[
2 - x^2 > 0.
\]

Answer.
The roots of the equation $2-x^2=0$ are $x = \pm \sqrt{2}$, leading to the following picture:

![Graph of positive and negative intervals](image)

The set of solutions of the inequality is the interval $(-\sqrt{2}, \sqrt{2})$. 
Exercise 2. Find the solutions of the inequality \( x^3 < x \).

**Answer.** Rewrite as \( x^3 - x < 0 \). The equation \( x^3 - x = 0 \) has as solutions \( x = -1, x = 0 \) and \( x = 1 \):

The set of solutions of the inequality is the set \((-\infty, -1) \cup (0,1)\).

Exercise 3. Find the solutions of the inequality

\[ x^3 + x^2 - 6x > 0. \]

**Answer.** The equation \( x^3 + x^2 - 6x = 0 \) has the roots \( x = -3, x = 0 \) and \( x = 2 \).

Consequently, the set of solutions of the inequality is the union of the interval \((-3, 0)\) and the interval \((2, \infty)\).

Exercise 4. Find the solutions of the inequality

\[ x^2 + 5x \geq 3. \]

**Answer.** By using the quadratic formula, we find that the equation \( x^2 + 5x - 3 = 0 \) has as its roots

\[ x = \frac{-5 \pm \sqrt{37}}{2} \]

Note that

\[ -\frac{5 - \sqrt{37}}{2} > 0, \]

while

\[ -\frac{5 + \sqrt{37}}{2} < 0. \]

The solutions are all points in the set

\[ (\infty, -\frac{5 + \sqrt{37}}{2}) \cup \left(-\frac{5 - \sqrt{37}}{2}, \infty\right). \]
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Exercise 5. Find the solutions of the inequality

\[ x^3 > x^2. \]

Answer.

Rewrite as \( x^3 - x^2 \geq 0 \). The roots of the corresponding equation are \( x=0 \) and \( x=1 \).

The tricky part is reading off all solutions: Clearly, the numbers in the interval \([1, \infty)\) are solutions, but don’t forget \( x=0 \). (Check that \( x=0 \) is indeed a solution!) Thus the set of solutions is the following set:

\[ \{0\} \cup [1, \infty). \]
Content Review for Section 2: Rational Expression

With regular fractions, multiplying and dividing is fairly simple, and is much easier than adding and subtracting. The situation is much the same with rational expressions (that is, with polynomial fractions). The only major problem I have seen students having with multiplying and dividing rationals is with illegitimate cancelling, where they try to cancel terms instead of factors, so I'll be making a big deal about that as we go along.

Multiplying Rational Expressions

Remember how you multiply regular fractions: You multiply across the top and bottom. For instance:

\[
\frac{3}{5} \times \frac{10}{9} = \frac{3 \times 10}{5 \times 9} = \frac{30}{45}
\]

And you need to simplify, whenever possible:

\[
\frac{30}{45} = \frac{2 \times 15}{3 \times 15} = \frac{2 \times 15}{3 \times 15} = \frac{2}{3}
\]

While the above simplification is perfectly valid, it is generally simpler to cancel first and then do the multiplication, since you'll be dealing with smaller numbers that way. In the above example, the 3 in the numerator of the first fraction duplicates a factor of 3 in the denominator of the second fraction, and the 5 in the denominator of the first fraction duplicates a factor of 5 in the numerator of the second fraction. Since anything divided by itself is just 1, we can "cancel out" these common factors (that is, we can ignore these forms of 1) to find a simpler form of the fraction:

\[
\frac{3}{5} \times \frac{10}{9} = \frac{2 \times 10}{3 \times 9} = \frac{2}{3}
\]

This process (cancelling first, then multiplying) works with rational expressions, too.

- Simplify the following expression:

\[
\frac{7x^2}{14x} \cdot \frac{9}{3}
\]

Simplify by cancelling off duplicate factors:

\[
\frac{7x^2}{14x} \cdot \frac{9}{3} = \frac{1 \times x^2}{2 \times x} \cdot \frac{3}{1} = \frac{x^2}{2} \cdot \frac{3}{1} = \frac{3x}{2}, \ x \neq 0
\]

If you're not sure how the variable parts were simplified above, you may want to review how to simplify expressions with exponents.

Then the answer is:

\[
\frac{3x}{2}, \ x \neq 0
\]
Why did I add the "for \( x \) not equal to 0" notation after the simplified fraction? Because the original expression was not defined at \( x = 0 \) (since this would have caused division by zero in the second of the two original fractions). For the two expressions, the original one and the simplified one, to be "equal" in technical terms, their domains have to be the same; they have to be defined for the same \( x \)-values. Since the simplified fraction, \( \frac{3x}{2} \), has no division-by-zero problem at \( x = 0 \), it is not, strictly-speaking, "equal" to the original expression. To make the simplified form truly equal to the original form, I have to explicitly state this "\( x \) cannot be zero" exclusion.

Multiply and simplify the following:

\[
15 \cdot \frac{4x + 5}{6}
\]

Many students find it helpful to convert the "15" into a fraction. This can make the factors a little more obvious, so one can see more clearly what can cancel with what.

\[
\frac{15}{1} \cdot \frac{4x + 5}{6} = \frac{5}{1} \cdot \frac{4x + 5}{\frac{6}{2}}
\]

Can I cancel off the 2 into the 20? No! When I have a fraction like this, there are understood parentheses around any sums of terms, like this:

\[
\frac{(20x + 25)}{2}
\]

I can only cancel off factors (the entire contents of a set of parentheses); I can NOT cancel terms (part of what is inside a set of parentheses).

The only thing that I can factor out of the \( 20x + 25 \) in the numerator is a 5, and that factor doesn't cancel off with the 2 underneath, so, for this rational, there is no further reduction to be done. Then my final answer is:

\[
\frac{20x + 25}{2}
\]
Multiply and simplify the following expression:

\[
\frac{x^2 + 4x + 3}{2x^2 - x - 10} \cdot \frac{2x^2 + 4x^3}{x^2 + 3x} \cdot \frac{x}{x^2 + 3x + 2}
\]

Some students, when faced with this problem, will do something like this:

\[
\frac{x^2 + 4x + 3}{2x^2 - x - 10} \cdot \frac{2x^2 + 4x^3}{x^2 + 3x} \cdot \frac{x}{x^2 + 3x + 2}
\]

You can not cancel terms; you can only cancel factors.

Since I can only cancel factors, my first step in this simplification has to be to factor all the numerators and denominators. Once I've factored everything, I can cancel off any factor that is mirrored on the two sides of the fraction line. The legitimate simplification looks like this:

\[
\frac{(x + 3)(x + 1)}{(2x - 5)(x + 2)} \cdot \frac{2x^2(1 + 2x)}{x(x + 3)(x + 1)} = \frac{x}{(2x - 5)(x + 2)^2}
\]

Can I now cancel off some 2's? No! The x's are only part of their respective factors; they are not stand-alone factors, so they can't cancel off with anything. Then my answer, taking note of the trouble-spots (the division-by-zero problems) that I removed when I cancelled the common factors, is:

\[
\frac{2x^2(1 + 2x)}{(2x - 5)(x + 2)^2}, \quad x \neq 0, -1, -3
\]

The "x not equal to 0, –1 or –3" came from the factors that I cancelled off; your book may not require this information as part of your answer.

Note: For reasons which will become clear when you are adding rational expressions, it is customary to leave the denominator factored, as shown above. At this stage, your book may or may not want the numerator factored. You should recognize, in any case, that "(2x – 5) (x + 2)^2"
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is the same thing as "$2x^3 + 3x^2 – 12x – 20"$, and get in the habit of converting the form of your answer if your book or instructor expects it written in a particular way.

Dividing Rational Expressions

For dividing rational expressions, you will use the same method as you used for dividing numerical fractions: when dividing by a fraction, you flip-n-multiply. For instance:

- **Perform the indicated operation:** \[ \frac{4}{3} - \frac{9}{5} \]

  To simplify this division, I'll convert it to multiplication by flipping what I'm dividing by; that is, I'll switch from dividing by a fraction to multiplying by that fraction's reciprocal. Then I'll simplify as usual:

  \[
  \frac{4}{3} \div \frac{9}{5} = \frac{4}{3} \cdot \frac{5}{9} = \frac{20}{27}
  \]

  Can the 2's cancel off from the 20's? No! This is as simplified as the fraction gets.

Division works the same way with rational expressions.

- **Perform the indicated operation:** \[ \frac{x^2+2x-15}{x^2-4x-45} - \frac{x^2+x-12}{x^2-5x-36} \]

  To simplify this, first I'll flip-n-multiply. Then, to simplify the multiplication, I'll factor the numerators and denominators, and then cancel any duplicated factors. My work looks like this:

  \[
  \frac{x^2+2x-15}{x^2-4x-45} \cdot \frac{x^2+x-12}{x^2-5x-36} = \frac{x^2+2x-15}{x^2-4x-45} \cdot \frac{x^2-5x-36}{x^2+x-12}
  \]

  \[
  = \frac{(x+5)(x-3)}{(x-9)(x+5)} \cdot \frac{(x-9)(x+4)}{(x+4)(x-3)}
  \]

  \[
  = \frac{(x+5)(x-3)}{(x-9)(x+5)} \cdot \frac{(x-9)(x+4)}{(x+4)(x-3)}
  \]

  Then the answer is: $1, x \neq -5, -4, 3, 5$
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Your instructor may not require the restrictions on the allowable \(x\)-values, in which case your answer would be just "1". The exercises you'll be given won't usually simplify this much, of course. The following example is much more typical:

- **Simplify the following expression:** \[
\frac{x^2 + 3x - 40}{x^2 + 2x - 35} \div \frac{x^2 + 2x - 48}{x^2 + 3x - 18}\]

First, I'll need to flip the second fraction, and convert from division to multiplication. Then I'll factor, and see if anything cancels.

\[
\frac{x^2 + 3x - 40}{x^2 + 2x - 35} \times \frac{x^2 + 2x - 48}{x^2 + 3x - 18} = \frac{(x + 8)(x - 5)}{(x + 7)(x - 6)} \times \frac{(x + 7)(x - 5)}{(x + 6)(x - 3)}
\]

Then the final answer is:

\[
\frac{(x + 6)(x - 3)}{(x + 7)(x + 6)} = \frac{x^2 + 3x - 18}{(x + 7)(x - 6)}, x \neq -8, -6, 5, 3
\]

For reasons which will become clear when adding and subtracting rationals, the numerator is usually multiplied out ("simplified" to get rid of the parentheses), while the denominator is usually left in factored form.

Make sure you know how to factor quadratics and cubics, because, as you have seen, it is required for many of the problems you'll be doing. Also, make sure you are careful to cancel only factors, not terms. If you can keep that straight, then you'll probably do fine.

**Proportions and Proportional Reasoning**

There are several ways to express a "ratio". Let's compare the number of boys with the number of girls in a particular classroom. Let's say that our classroom has 25 students, 10 of whom are boys. That means there are 15 girls. So, the ratio of boys to girls is 10 to 15.
Writing a Ratio

There are three ways of expressing a ratio. You can simply use words like we did above, or you can separate the two numbers using a colon or a fraction (the mathematician's choice).

10:15 or 10/15

In mathematics, we always use fractions to represent ratios. In this classroom example, many other ratios that can be made:

\[ \frac{10}{25} \text{ the number of boys out of the total number of students} \]

\[ \frac{15}{25} \text{ the number of girls out of the total number of students} \]

\[ \frac{15}{10} \text{ the number of girls to the number of boys} \]

Simplifying a Ratio

Since we're using a fraction to represent ratios, the ratios can sometimes be reduced. For example, the ratio of boys to girls in our hypothetical classroom is \( \frac{10}{15} \), but can be reduced to \( \frac{2}{3} \). So, if we would be correct we said that the ratio of boys to girls in our hypothetical classroom is \( 2:3 \), or two boys for every 3 girls.

Proportions

The following equation is a proportion:

\[ \frac{10}{15} = \frac{2}{3} \]

Any proportion is simply an equation the states two ratios are equal to one another. Sometimes, a proportion may contain variables:

\[ \frac{2}{3} = \frac{x}{30} \]

If we wish to solve such an equation, we can use a process called cross-multiplication.

Solving Proportions using Cross-Multiplication

If \( \frac{a}{b} = \frac{c}{d} \), then the products that are formed by diagonals across the equal sign are also equal: \( ad = bc \). Note that this would be equivalent to saying \( bc = ad \).
Solving Rational Equations

While adding and subtracting rational expressions is a royal pain, solving rational equations is much simpler. (Note that I don't say that it's "simple", just that it's "simpler".) This is because, as soon as you go from a rational expression (with no "equals" sign in it) to a rational equation (with an "equals" sign in the middle), you get a whole different set of tools to work with. In particular, you can multiply through on both sides of the equation to get rid of the denominators.

- Solve the following equation: \( \frac{2}{3} = \frac{x}{3} \)

This equation is so simple that I can solve it just by looking at it: since I have two-thirds equal to \( x \)-thirds, clearly \( x = 2 \). The reason this was so easy to solve is that the denominators were the same, so all I had to do was solve the numerators. \( x = 2 \)

- Solve the following equation: \( \frac{x-1}{15} = \frac{2}{5} \)

To solve this, I can convert to a common denominator of 15:

\[
\frac{x-1}{15} = \left( \frac{2}{5} \right) \left( \frac{3}{3} \right)
\]

\[
\frac{x-1}{15} = \frac{6}{15}
\]

Now I can compare the numerators:

\[
x - 1 = 6
\]

\[
x = 7
\]

Note, however, that I could also have solved this by multiplying through on both sides by the common denominator:

\[
\left( \frac{x-1}{15} \right) \left( \frac{15}{1} \right) = \left( \frac{2}{5} \right) \left( \frac{15}{1} \right)
\]

\[
\frac{x-1}{15} = \frac{2(15)}{5}
\]

\[
x - 1 = 2(3)
\]

\[
x - 1 = 6
\]

\[
x = 7
\]

When you were adding and subtracting rational expressions, you had to find a common denominator. Now that you have equations (with an "equals" sign in the middle), you are
allowed to multiply through (because you have two sides to multiply on) and get rid of the denominators entirely. In other words, you still need to find the common denominator, but you don't necessarily need to use it in the same way.

Here are some more complicated examples:

- **Solve the following equation:** \( \frac{3}{x+2} - \frac{1}{x} = \frac{1}{5x} \)

First, I need to check the denominators: they tell me that \( x \) cannot equal zero or \(-2\) (since these values would cause division by zero). I'll re-check at the end, to make sure any solutions I find are "valid".

There are two ways to proceed with solving this equation. I could convert everything to the common denominator of \( 5x(x+2) \) and then compare the numerators:

\[
\left( \frac{3}{x+2} \right) \left( \frac{5x}{5x} \right) - \left( \frac{1}{x} \right) \left( \frac{5x(x+2)}{5x(x+2)} \right) = \left( \frac{1}{5x} \right) \left( \frac{x+2}{x+2} \right)
\]

\[
\frac{15x}{5x(x+2)} - \frac{5x+10}{5x(x+2)} = \frac{x+2}{5x(x+2)}
\]

At this point, the denominators are the same. So do they really matter? Not really (other than for saying what values \( x \) can't be). At this point, the two sides of the equation will be equal as long as the numerators are equal. That is, all I really need to do now is solve the numerators:

\[
15x - (5x + 10) = x + 2
\]
\[
10x - 10 = x + 2
\]
\[
9x = 12
\]
\[
x = \frac{12}{9} = \frac{4}{3}
\]

Since \( x = \frac{4}{3} \) won't cause any division-by-zero problems in the fractions in the original equation, then this solution is valid. \( x = \frac{4}{3} \)

I said there were two ways to solve this problem. The above is one method. Another method is to find the common denominator but, rather than converting everything to that denominator, I'll take advantage of the fact that I have an equation here, and multiply through on both sides by that common denominator. This will get rid of the denominators:

\[
\left( \frac{3}{x+2} \right) \left( \frac{5x(x+2)}{1} \right) - \left( \frac{1}{x} \right) \left( \frac{5x(x+2)}{1} \right) = \left( \frac{1}{5x} \right) \left( \frac{5x(x+2)}{1} \right)
\]

\[
\left( \frac{3}{x+2} \right) \left( \frac{5x(x+2)}{1} \right) - \left( \frac{1}{x} \right) \left( \frac{5x(x+2)}{1} \right) = \left( \frac{1}{5x} \right) \left( \frac{5x(x+2)}{1} \right)
\]
To solve this, I can convert to a common denominator of 15:

\[
\frac{x-1}{15} = \frac{2}{3}, \quad \frac{x-1}{15} = \frac{6}{15}
\]

Now I can compare the numerators:

\[
x - 1 = 6
\]

\[
x = 7
\]

Note, however, that I could also have solved this by multiplying through on both sides by the common denominator:

\[
\frac{x-1}{15} = \frac{2}{3}, \quad \frac{x-1}{15} = \frac{6}{15}
\]

\[
x - 1 = 2(3)
\]

\[
x - 1 = 6
\]

\[
x = 7
\]

When you were adding and subtracting rational expressions, you had to find a common denominator. Now that you have equations (with an "equals" sign in the middle), you are allowed to multiply through (because you have two sides to multiply on) and get rid of the denominators entirely. In other words, you still need to find the common denominator, but you don't necessarily need to use it in the same way.

Here are some more complicated examples:

**Solve the following equation:**

\[
\frac{3}{x+2} - \frac{1}{x} = \frac{1}{5x}
\]

First, I need to check the denominators: they tell me that \(x\) cannot equal zero or \(-2\) (since these values would cause division by zero). I'll re-check at the end, to make sure any solutions I find are "valid".
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There are two ways to proceed with solving this equation. I could convert everything to the common denominator of $5x(x + 2)$ and then compare the numerators:

$$\left( \frac{3}{x+2} \right) \left( \frac{5x}{5x} \right) - \left( \frac{1}{x} \right) \left( \frac{5x(x+2)}{5x} \right) = \left( \frac{1}{5x} \right) \left( \frac{x+2}{x+2} \right)$$

$$\frac{15x}{5x(x+2)} - \frac{5x+10}{5x(x+2)} = \frac{x+2}{5x(x+2)}$$

At this point, the denominators are the same. So do they really matter? Not really (other than for saying what values $x$ can't be). At this point, the two sides of the equation will be equal as long as the numerators are equal. That is, all I really need to do now is solve the numerators:

$$15x - (5x + 10) = x + 2$$
$$10x - 10 = x + 2$$
$$9x = 12$$
$$x = \frac{12}{9} = \frac{4}{3}$$

Since $\frac{4}{3}$ won't cause any division-by-zero problems in the fractions in the original equation, then this solution is valid.

I said there were two ways to solve this problem. The above is one method. Another method is to find the common denominator but, rather than converting everything to that denominator, I'll take advantage of the fact that I have an equation here, and multiply through on both sides by that common denominator. This will get rid of the denominators:

$$\left( \frac{3}{x+2} \right) \left( \frac{5x(x+2)}{1} \right) - \left( \frac{1}{x} \right) \left( \frac{5x(x+2)}{1} \right) = \left( \frac{1}{5x} \right) \left( \frac{5x(x+2)}{1} \right)$$

$$\frac{3(5x)}{x+2} - \frac{5x(x+2)}{x} = \frac{5x(x+2)}{5x}$$

$$3(5x) - 1(5x + 2) = 1(x + 2)$$
$$15x - 5x - 10 = x + 2$$
$$10x - 10 = x + 2$$
$$9x = 12$$
$$x = \frac{12}{9} = \frac{4}{3}$$

This method gives the same result as the first method. I view this second method as being quicker and easier, but this is only my personal preference. My students have typically been fairly evenly divided in their preferences for the two methods. I will do each of the examples in the following pages both ways. You should pick the method that works best for you.
Solve the following equation:  \( \frac{10}{x(x-2)} + \frac{4}{x} = \frac{5}{x-2} \)

The common denominator here will be \( x(x-2) \), and \( x \) cannot be zero or 2. I can solve this equation by multiplying through on both sides of the equation by this denominator:

\[
\left( \frac{10}{x(x-2)} \right) \left( \frac{x(x-2)}{1} \right) + \left( \frac{4}{x} \right) \left( \frac{x(x-2)}{1} \right) = \left( \frac{5}{x-2} \right) \left( \frac{x(x-2)}{1} \right)
\]

\[
\frac{10}{x(x-2)} \left( x(x-2) \right) + \frac{4}{x} \left( x(x-2) \right) = \frac{5}{x-2} \left( x(x-2) \right)
\]

\[
10 + 4(x-2) = 5x
\]

\[
10 + 4x - 8 = 5x
\]

\[
4x + 2 = 5x
\]

\[
x = 2
\]

Or I can solve by converting to the common denominator and then solving the numerators:

\[
\frac{10}{x(x-2)} + \left( \frac{4}{x} \right) \left( \frac{x-2}{x-2} \right) = \left( \frac{5}{x-2} \right) \left( \frac{x}{x} \right)
\]

\[
\frac{10}{x(x-2)} + \frac{4x-8}{x(x-2)} = \frac{5x}{x(x-2)}
\]

\[
10 + (4x - 8) = 5x
\]

\[
10 + 4x - 8 = 5x
\]

\[
4x + 2 = 5x
\]

\[
x = 2
\]

Either way, the answer I get is: \( x = 2 \)

However, I need to check this solution with the original equation. Do you see that I'm going to have a problem with \( x = 2 \)? This value would cause division by zero in the original equation! Since the only possible solution causes division by zero, then this equation really has no solution.

How did we end up with an invalid solution? We didn't do anything wrong. But notice that, whichever method you use to solve a rational equation, at some point you're going to get rid of the denominators. For rational equations, the difficulties come from those denominators. So whenever you solve a rational, always check the solution against the denominators in the original problem. It is entirely possible (not commonly in homework, but almost always on the test) that a problem will have an invalid ("extraneous") solution.
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- Solve the following equation: \( \frac{x}{x-2} + \frac{x}{x-4} = \frac{2}{x^2-6x+8} \)

First I'll need to factor that quadratic, so I can tell what factors I'll have in my common denominator.

\[ x^2 - 6x + 8 = (x - 4)(x - 2) \]

That worked out nicely: the factors of the quadratic are duplicates of the other denominators. (This often happens for these problems.) So the common denominator will be \((x - 4)(x - 2)\), and I'll need to remember (at the end) that \(x\) cannot be 2 or 4.

I can convert everything to the common denominator and then solve the numerators:

\[
\frac{x}{x-2} \cdot \frac{x-4}{x-4} + \frac{1}{x-4} \cdot \frac{x-2}{x-2} = \frac{2}{(x-2)(x-4)}
\]

\[
x^2 - 4x + x - 2 = 2
\]

\[
x^2 - 3x - 4 = 0
\]

\[
(x - 4)(x + 1) = 0
\]

\[
x = 4 \text{ or } x = -1
\]

...or I can multiply through on both sides by the common denominator and solve the resulting equation:

\[
\frac{x}{x-2} \cdot \frac{x-4}{1} + \frac{1}{x-4} \cdot \frac{x-2}{1} = \frac{2}{(x-2)(x-4)}
\]

\[
x(x-4) + 1(x-2) = 2
\]

\[
x^2 - 4x + x - 2 = 2
\]

\[
x^2 - 3x - 2 = 2
\]

\[
x^2 - 3x - 4 = 0
\]

\[
(x - 4)(x + 1) = 0
\]

\[
x = 4 \text{ or } x = -1
\]

Either way, I get the same result: \(x = 4\) and \(x = -1\). Checking these solutions against the denominators of the original equation, I see that "\(x = 4\)" would cause division by zero, so I throw that solution out. Then the answer is:

\[ x = -1 \]
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- **Solve the following equation:** 
  \[ x + 1 = \frac{72}{x} \]

There is only one fraction, so the common denominator is just \( x \), and the solution cannot be \( x = 0 \). To solve, I can start by multiplying through on both sides by \( x \):

\[
\begin{align*}
(x)(x) + (1)(x) &= \frac{72}{x} \\
(x)(x) + (1)(x) &= \frac{72}{x}
\end{align*}
\]

\[
\begin{align*}
x^2 + x &= 72 \\
x^2 + x - 72 &= 0 \\
(x + 9)(x - 8) &= 0 \\
x &= -9 \text{ or } x = 8
\end{align*}
\]

...or I can convert to the common denominator and solve the numerators:

- **Solve the following equation:** 
  \[ x + 1 = \frac{72}{x} \]

There is only one fraction, so the common denominator is just \( x \), and the solution cannot be \( x = 0 \). To solve, I can start by multiplying through on both sides by \( x \):

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x^2 + x &= 72 \\
x^2 + x - 72 &= 0 \\
(x + 9)(x - 8) &= 0 \\
x &= -9 \text{ or } x = 8
\end{align*}
\]

...or I can convert to the common denominator and solve the numerators:

\[
\begin{align*}
\left(\frac{x}{x}\right) + \left(\frac{1}{x}\right) &= \frac{72}{x} \\
\left(\frac{x}{x}\right) + \left(\frac{1}{x}\right) &= \frac{72}{x}
\end{align*}
\]

\[
\begin{align*}
x^2 + x &= 72 \\
x^2 + x - 72 &= 0 \\
(x + 9)(x - 8) &= 0 \\
x &= -9 \text{ or } x = 8
\end{align*}
\]

Either way, the solution is \( x = -9 \) or \( x = 8 \). Since neither solution causes a division-by-zero problem in the original equation, both solutions are valid.

\[ x = -9 \text{ or } x = 8 \]
• Solve the following equation:

\[
\frac{10}{x+4} = \frac{15}{4(x+1)}
\]

First, I note that I cannot have \(x = -1\) or \(x = -4\). Then I notice that this equation is a proportion: the equation is of the form "one fraction equals another fraction". So all I need to do here is cross-multiply.

\[
10(4(x + 1)) = 15(x + 4)
\]

\[
40x + 40 = 15x + 60
\]

\[
25x = 20
\]

\[
x = \frac{20}{25} = \frac{4}{5}
\]

Since this solution won't cause any division-by-zero problems, it is valid: \(x = 4/5\)

As an aside, you may find it useful to remember that, when solving equations like these, you are (technically) trying to find the intersections of the functions on either side of the "equals" sign. For instance, let's return to this equation:

\[
\frac{x}{x-2} + \frac{1}{x-4} = \frac{2}{x^2 - 6x + 8}
\]

Let either side of the equation be its own function:

\[
y_1 = \frac{x}{x-2} + \frac{1}{x-4}
\]

\[
y_2 = \frac{2}{x^2 - 6x + 8}
\]

Graphing these, you can see that they intersect in one spot:

This is the solution, \(x = -1\), that we found earlier. But remember how we initially came up with two solutions? That was because we'd gotten to where we were ignoring the denominators. If we do that, we get the following functions:

\[
y_3 = x^2 - 3x - 2
\]

\[
y_4 = 2
\]

These graph like this:

As you can see, getting rid of the denominators create
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If you have a graphing calculator, by the way, keep in mind that a quick graph can allow you to check your answers (since solutions to the equations are intersections of the graphs) before you hand in your test. Just make sure your solutions match the x-values of the intersection points on the graphs.

Solving rational equations is pretty straightforward if you are careful to write each step completely. But (warning!) as soon as you start skipping steps or doing stuff in your head, you’re going to start messing up. So always work neatly and completely. And never forget to check your solutions, because I can just about guarantee that you’ll have one of those "no solution" (or "only one solution works") problems on your test.

*Continuous Functions and Discontinuities*

A function that is **continuous** is a function whose graph has no breaks in it; i.e. it is a continuous curve. Generally speaking, a function is **continuous** if you can draw its graph without picking up your pencil. Notice, on the graph of \( y = \sin(x) \), that the function is completely connected at all points. Many functions, however, will have isolated points where they are not connected. These problem points are called **discontinuities**. There are three types of discontinuities:

**Discontinuity 1: Asymptotic Discontinuities**

In the function \( f(x) = \frac{x+4}{(x-1)(x+8)} \) we know that the domain is limited to all real numbers except 1 and -8. Often, the most interesting points in a function are the problematic points, and indeed, we can see in the graph that the function behaves very strangely at the holes in the domain.

It may be revealing to look at the values of the function approaching one of the problematic points, \( x = 1 \):

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>-0.687</td>
</tr>
<tr>
<td>0.5</td>
<td>-1.059</td>
</tr>
<tr>
<td>0.75</td>
<td>-2.171</td>
</tr>
<tr>
<td>1.0</td>
<td>undefined</td>
</tr>
<tr>
<td>1.25</td>
<td>2.270</td>
</tr>
<tr>
<td>1.5</td>
<td>1.158</td>
</tr>
<tr>
<td>1.75</td>
<td>0.786</td>
</tr>
</tbody>
</table>
As we approach x = 1 from the left side, with increasing x values, the y value drops to lower and lower negatives. The graph shows us that the y value seems to approach - ∞ from the left as the x-value approaches x = 1, marked by the dotted line. This represents a discontinuity, since the function is not connected over the dotted line. Specifically, this type of discontinuity is called an **asymptotic discontinuity**. The dotted lines represent asymptotes; they are values for which the function never takes a value, yet still approaches. The asymptotes we see with this function are called **vertical asymptotes** because they are vertical lines. There are also **horizontal asymptotes**. To see horizontal asymptotes, try graphing the function \( f(x) = \frac{2x}{x+4} \).

It should have a horizontal asymptote at y = 2. In order to figure out where horizontal asymptotes occur in a function, check out the section on limits.

In general, **asymptotes occur when a function approaches infinity at a specific value of x or y**. If a function has values on both sides of an asymptote, then it cannot be connected, so it must have a discontinuity at the asymptote. **We look for asymptotes at points where the denominator is zero because** when the denominator gets close to zero and becomes very small, it makes the value of the function very large. For example, if we look at the fraction \( \frac{5}{0.1} \), i.e. divide 5 by 0.1, we get that it equals 50. If we make the denominator smaller, the value of the fraction gets larger: \( \frac{5}{0.01} = 500, \frac{5}{0.00001} = 500000 \). This tells us that the closer to zero that the denominator is, the larger the value of the fraction. In other words, we've demonstrated
\[
\lim_{x \to 0} \frac{1}{x} = \infty
\]

**Discontinuity 2: Point Discontinuities**

Point discontinuities are also called **removable discontinuities** or **removable singularities**.

Sometimes we come across functions that are defined differently for a certain point. Consider the function \( f(x) = \begin{cases} 1, & x = 3 \\ x^2, & \text{all other real } x \end{cases} \). We defined the value of the function to be 1 at the point x = 3, yet, the rest of the function is dictated by \( f(x) = x^2 \). We can see in the graph that the function is continuous except for the tiny hole in the curve at x = 1. It is discontinuous at a single point, and this discontinuity is called a **point discontinuity**.

In general, **point discontinuities occur when a function is defined specifically for an isolated x-value**. However, this does not guarantee a point discontinuity. For example, if we change our function slightly to \( f(x) = \begin{cases} 1, & x = 3 \\ x^2, & \text{all other real } x \end{cases} \) it becomes continuous. This is because we have defined the value of the function at \( f(3) \) precisely to be the value of the function \( f(x) = x^2 \) at \( x = 3 \). In this case, we did not define the value at \( x = 3 \) to be different from what it would be if the function were \( f(x) = x^2 \). Then there is no discontinuity. Compared to our last function with a point discontinuity, we moved the point back up to the function to
"plug" up the hole, and it is now continuous. Always remember, if a function is defined like this, to check if the isolated point is a point discontinuity or just a trick.

Point Discontinuities also arise when our function has a denominator that can be equal to zero, but that part of the denominator can also be cancelled out with a like term in the numerator. Consider the function \( f(x) = \frac{x^2(x-2)}{x-2} \). If we try to find the value of the function at \( x = 2 \), we end up getting \( f(2) = \frac{2^2(2-2)}{2-2} = \frac{0}{0} \). 0/0 represents an undefined number - i.e. the function does not exist at that point. However, if we restrict the function to a domain that does not include \( x = 2 \), we can simply cancel out the \( \frac{(x-2)}{(x-2)} \) and be left with \( f(x) = x^2 \). This leaves us to define the function as
\[
 f(x) = \begin{cases} 
 x^2, & x \neq 2 \\
 \text{undefined}, & x = 2 
\end{cases}
\] We have effectively removed the discontinuity to show that the function behaves exactly like \( f(x) = x^2 \), except at \( x = 2 \), where it is undefined.

In conclusion, point discontinuities also occur when we can cancel a term in the denominator and the numerator. They occur at the values for which the cancelled term is equal to zero. In our example, we removed \( x - 2 \), and \( x - 2 = 0 \) at \( x = 2 \). If we were to remove \( \sin(x) \), we would have point discontinuities at integer multiples of \( \pi \), since \( \sin(\pi) = \sin(2\pi) = \sin(3\pi) = \sin(n\pi) = 0 \) for any integer \( n \).

**Discontinuity 3: Jump Discontinuities**

Jump discontinuities are also called simple discontinuities, or continuities of the first kind. Just as we can define a function at a specific point, we can also define a function in specific regions. Consider the function
\[
 f(x) = \begin{cases} 
 x^2, & x \leq 1 \\
 2 - x, & x > 1 
\end{cases}
\] For all intents and purposes, we can consider it two separate functions - one that is defined for \( x \) less than or equal to 1 and another function that is defined for \( x \) greater than 1. It is useful to think of it this way when we eventually use integration, differentiation, and other such mathematical tools. However, as it is written, \( f \) is a single function, called a piecewise function, since it is defined piece-by-piece. Note that the function adheres to our definition of being continuous.

Now, suppose we change our function slightly to
\[
 f(x) = \begin{cases} 
 x^2, & x \leq 1 \\
 6 - x, & x > 1 
\end{cases}
\] The two pieces now have a different value at \( x = 1 \), and we can see in the graph that our function \( f \) seems to "jump" from one branch to the other. Note this this jump makes the function discontinuous. We refer to this as a jump discontinuity.
Notice that the function's discontinuity is entirely dependent on the value of the two branches of the function. Because of this, we can't just look at a piecewise function and immediately see if there is a jump discontinuity. Look at the following animation, which essentially moves the linear branch of the piecewise function down, and note that it is only continuous at one point.

In this demonstration, we graph the function \( f(x) = \begin{cases} x^2, & x \leq 1 \\ c - x, & x > 1 \end{cases} \) and the animation changes the value of \( c \).

Jump discontinuities occur where the function approaches two different values from either side of the discontinuity. In our example, on the right side of \( x = 1 \), the function is approaching the value \( f(1) = 5 \). On the left side of \( x = 1 \), the function is approaching the value \( f(1) = 1 \). Thus, it has a jump discontinuity. Formally, we can check this by checking if the left-hand limit and the right-hand limit of the function correspond to the same value at a given point. For more information about left-hand and right-hand limits, please check out the limits page.

Example 1

We have to check the following function for discontinuities: \( f(x) = \begin{cases} \frac{x+4}{x}, & x \leq 2 \\ x^3 + 1, & x > 2 \end{cases} \)

We begin by looking at the first branch of the function. First, note that it has an \( x \) in its denominator. This tells us that it has a problem in its domain at \( x = 0 \). As the values for \( x \) get very small, the value of the function approaches infinity, so we have an asymptotic discontinuity at \( x = 0 \). We also have to check if the branches correspond to the same value at \( x = 2 \). The first branch yields \( \frac{2+4}{2} = 3 \) and the second branch yields \( 2^3 + 1 = 9 \). We can see the functions approach different values at \( x = 2 \) so there is a jump discontinuity at \( x = 2 \). There are no other discontinuities.
Content Review for Section 3: Radical Expression and Equations

❖ Introduction & Simplification

"Roots" (or "radicals") are the "opposite" operation of applying exponents; you can "undo" a power with a radical, and a radical can "undo" a power. For instance, if you square 2, you get 4, and if you "take the square root of 4", you get 2; if you square 3, you get 9, and if you "take the square root of 9", you get 3:

\[ 2^2 = 4, \text{ so } \sqrt{4} = 2 \]
\[ 3^2 = 9, \text{ so } \sqrt{9} = 3 \]

The "\( \sqrt{\)" symbol is called the "radical" symbol. (Technically, just the "check mark" part of the symbol is the radical; the line across the top is called the "vinculum"). The expression "\( \sqrt{9} \)" is read as "root nine", "radical nine", or "the square root of nine".

You can raise numbers to powers other than just 2; you can cube things, raise them to the fourth power, raise them to the 100th power, and so forth. In the same way, you can take the cube root of a number, the fourth root, the 100th root, and so forth. To indicate some root other than a square root, you use the same radical symbol, but you insert a number into the radical, tucking it into the "check mark" part. For instance:

\[ 4^3 = 64, \text{ so } \sqrt[3]{64} = 4 \]

The "3" in the above is the "index" of the radical; the "64" is "the argument of the radical", also called "the radicand". Since most radicals you see are square roots, the index is not included on square roots. While "\( \sqrt[2]{\)" would be technically correct, I've never seen it used.

a square (second) root is written as \( \sqrt{\)

a cube (third) root is written as \( \sqrt[3]{\)

a fourth root is written as \( \sqrt[4]{\)

a fifth root is written as: \( \sqrt[5]{\)

You can take any counting number, square it, and end up with a nice neat number. But the process doesn't always work going backwards. For instance, consider \( \sqrt[3]{3} \), the square root of
three. There is no nice neat number that squares to 3, so \( \sqrt{3} \) cannot be simplified as a nice whole number. You can deal with \( \sqrt{3} \) in either of two ways: If you are doing a word problem and are trying to find, say, the rate of speed, then you would grab your calculator and find the decimal approximation of \( \sqrt{3} \):

\[
\sqrt{3} \approx 1.732050808
\]

Then you'd round the above value to an appropriate number of decimal places and use a real-world unit or label, like "1.7 ft/sec". On the other hand, you may be solving a plain old math exercise, something with no "practical" application. Then they would almost certainly want the "exact" value, so you'd give your answer as being simply "\( \sqrt{3} \)".

\[ \text{Simplifying Square-Root Terms} \]

To simplify a square root, you "take out" anything that is a "perfect square"; that is, you take out front anything that has two copies of the same factor:

\[
\sqrt{4} = \sqrt{2^2} = 2
\]

\[
\sqrt{19} = \sqrt{7^2} = 7
\]

\[
\sqrt{225} = \sqrt{15^2} = 15
\]

Note that the value of the simplified radical is positive. While either of +2 and –2 might have been squared to get 4, "the square root of four" is defined to be only the positive option, +2. When you solve the equation \( x^2 = 4 \), you are trying to find all possible values that might have been squared to get 4. But when you are just simplifying the expression \( \sqrt{2} \), the ONLY answer is "2"; this positive result is called the "principal" root. (Other roots, such as –2, can be defined using graduate-school topics like "complex analysis" and "branch functions", but you won't need that for years, if ever.)

Sometimes the argument of a radical is not a perfect square, but it may "contain" a square amongst its factors. To simplify, you need to factor the argument and "take out" anything that is a square; you find anything you've got a pair of inside the radical, and you move it out front. To do this, you use the fact that you can switch between the multiplication of roots and the root of a multiplication. In other words, radicals can be manipulated similarly to powers:

\[
(ab)^n = a^n b^n \quad \text{and} \quad n\sqrt{ab} = \sqrt[n]{a} \sqrt[n]{b}
\]
• **Simplify** √144

There are various ways I can approach this simplification. One would be by factoring and then taking two different square roots:

\[ \sqrt{144} = \sqrt{9 \times 16} = \sqrt{9} \sqrt{16} = 3 \times 4 = 12 \]

The square root of 144 is **12**.

You probably already knew that 12² = 144, so obviously the square root of 144 must be 12. But my steps above show how you can switch back and forth between the different formats (multiplication inside one radical, versus multiplication of two radicals) to help in the simplification process.

• **Simplify** √24√6

Neither of 24 and 6 is a square, but what happens if I multiply them inside one radical?

\[ \sqrt{24 \times 6} = \sqrt{144} = \sqrt{12 \times 12} = 12 \]

• **Simplify** √75

This answer is pronounced as "five, root three". It is proper form to put the radical at the end of the expression. Not only is "√35" non-standard, it is very hard to read, especially when hand-written. And write neatly, because "5√3" is not the same as "√35".

You don't have to factor the radicand all the way down to prime numbers when simplifying. As soon as you see a pair of factors or a perfect square, you've gone far enough.

• **Simplify** √72

Since 72 factors as 2×36, and since 36 is a perfect square, then:

\[ \sqrt{72} = \sqrt{2 \times 36} = \sqrt{2 \times 6 \times 6} = 6\sqrt{2} \]

Since there had been only one copy of the factor 2 in the factorization 2×6×6, that left-over 2 couldn't come out of the radical and had to be left behind.

• **Simplify** √4500

\[ \sqrt{4500} = \sqrt{45 \times 100} = \sqrt{5 \times 9 \times 100} = 3 \times 10 \times \sqrt{5} = 30\sqrt{5} \]

Variables in a radical's argument are simplified in the same way: whatever you've got a pair of can be taken "out front".
• Simplify $\sqrt{16x^4}$

$$\sqrt{16x^4} = \sqrt{4 \times 4 \times x \times x} = 4 \times x = 4x^2$$

• Simplify $\sqrt{12a^4b^7c^3}$

The 12 is the product of 3 and 4, so I have a pair of 2’s but a 3 left over. Also, I have two pairs of $a$’s; three pairs of $b$’s, with one $b$ left over; and one pair of $c$’s, with one $c$ left over. So the root simplifies as:

$$\sqrt{12a^4b^7c^3} = \sqrt{3 \times 2 \times 2 \times a \times a \times b \times b \times b \times b \times c \times c \times c}$$

$$= 2 \times a \times b \times b \times c \times \sqrt{3 \times b \times c}$$

$$= 2a^2b^3 \sqrt{3bc}$$

You are used to putting the numbers first in an algebraic expression, followed by any variables. But for radical expressions, any variables outside the radical should go in front of the radical, as shown above.

• Simplify $\sqrt{20r^{18}s^{21}t^{21}}$

Writing out the complete factorization would be a bore, so I’ll just use what I know about powers. The 20 factors as 4×5, with the 4 being a perfect square. The $r^{18}$ has nine pairs of $r$’s; the $s$ is unpaired; and the $t^{21}$ has ten pairs of $t$’s, with one $t$ left over. Then:

$$\sqrt{20r^{18}s^{21}t^{21}} = \sqrt{4 \times r^{18} \times s^{20} \times s \times t}$$

$$= 2 \times r^{9} \times s^{10} \times \sqrt{s \times t}$$

Technical point: Your textbook may tell you to "assume all variables are positive" when you simplify. Why? The square root of the square of a negative number is not the original number. For instance, you could start with $-2$, square to get $+4$, and then take the square root (which is defined to be the positive root) to get $+2$. You plugged in a negative and ended up with a positive. Sound familiar? It should: it’s how the absolute value works: $|-2| = +2$. Taking the square root of the square is in fact the technical definition of the absolute value. But this technicality can cause difficulties if you’re working with values of unknown sign; that is, with variables. The $|−2|$ is $+2$, but what is the sign on $|\ x \ |$? You can’t know, because you don’t know the sign of $x$ itself — unless they specify that you should "assume all variables are positive", or at least non-negative (which means "positive or zero").

❖ Multiplying Square Roots

The first thing you'll learn to do with square roots is "simplify" terms that add or multiply roots.

Simplifying multiplied radicals is pretty simple. We use the fact that the product of two radicals is the same as the radical of the product, and vice versa.
Write as the product of two radicals: $\sqrt{6}$

$$\sqrt{6} = \sqrt{2 \times 3} = \sqrt{2} \sqrt{3}$$

Okay, so that manipulation wasn't very useful. But working in the other direction can be helpful:

- Simplify by writing with no more than one radical: $\sqrt{2} \sqrt{8}$

$$\sqrt{2} \sqrt{8} = \sqrt{2 \times 8} = \sqrt{16} = \sqrt{4 \times 4} = 4$$

- Simplify by writing with no more than one radical: $\sqrt{3} \sqrt{6}$

$$\sqrt{3} \sqrt{6} = \sqrt{3 \times 6} = \sqrt{18} = 3 \sqrt{2}$$

- Simplify by writing with no more than one radical: $\sqrt{6} \sqrt{15} \sqrt{10}$

$$\sqrt{6} \sqrt{15} \sqrt{10} = \sqrt{6 \times 15 \times 10} = \sqrt{900} = \sqrt{2 \times 3 \times 5 \times 2 \times 3 \times 5 \times 2} = 30 \sqrt{2}$$

The process works the same way when variables are included:

- Simplify by writing with no more than one radical: $\sqrt{4x} \sqrt{5x^3}$

$$\sqrt{4x} \sqrt{5x^3} = \sqrt{4 \times 5 \times x \times x \times x} = \sqrt{2 \times 2 \times 5 \times x \times x \times x} = 2 \sqrt{x^2} \sqrt{5} = 2x \sqrt{5}$$

**Adding/Subtracting Square Roots**

Just as with "regular" numbers, square roots can be added together. But you might not be able to simplify the addition all the way down to one number. Just as "you can't add apples and oranges", so also you cannot combine "unlike" radicals. To add radical terms together, they have to have the same radical part.

- Simplify: $2\sqrt{3} + 3\sqrt{3}$

Since the radical is the same in each term (namely, the square root of three), I can combine the terms. I have two copies of the radical, added to another three copies. This gives me five copies:

$$2\sqrt{3} + 3\sqrt{3} = (2 + 3) \sqrt{3} = 5\sqrt{3}$$
That middle step, with the parentheses, shows the reasoning that justifies the final answer. You probably won't ever need to "show" this step, but it's what should be going through your mind.

- **Simplify: \( \sqrt{3} + 4\sqrt{3} \)**

The radical part is the same in each term, so I can do this addition. To help me keep track that the first term means "one copy of the square root of three", I'll insert the "understood" "1":

\[
\sqrt{3} + 4\sqrt{3} = (1 + 4) \sqrt{3} = 5\sqrt{3}
\]

Don't assume that expressions with unlike radicals cannot be simplified. It is possible that, after simplifying the radicals, the expression can indeed be simplified.

- **Simplify: \( \sqrt{9} + \sqrt{25} \)**

To simplify a radical addition, I must first see if I can simplify each radical term. In this particular case, the square roots simplify "completely" (that is, down to whole numbers):

\[
\sqrt{9} + \sqrt{25} = 3 + 5 = 8
\]

- **Simplify: \( 3\sqrt{4} + 2\sqrt{4} \)**

I have three copies of the radical, plus another two copies, giving me— Wait a minute! I can simplify those radicals right down to whole numbers:

\[
3\sqrt{4} + 2\sqrt{4} = 3 \times 2 + 2 \times 2 = 6 + 4 = 10
\]

Don't worry if you don't see a simplification right away. If I hadn't noticed until the end that the radical simplified, my steps would have been different, but my final answer would have been the same:

\[
3\sqrt{4} + 2\sqrt{4} = 5\sqrt{4} = 5 \times 2 = 10
\]

- **Simplify: \( 3\sqrt{3} + 2\sqrt{5} + \sqrt{3} \)**

I can only combine the "like" radicals, so I'll end up with two terms in my answer:

\[
3\sqrt{3} + 2\sqrt{5} + \sqrt{3} = 3\sqrt{3} - 1\sqrt{3} + 2\sqrt{5} = 4\sqrt{3} + 2\sqrt{5}
\]

There is not, to my knowledge, any preferred ordering of terms in this sort of expression, so the expression \( 2\sqrt{5} + 4\sqrt{3} \) should also be an acceptable answer.

- **Simplify: \( 3\sqrt{8} + 5\sqrt{2} \)**

I can simplify the radical in the first term, and this will create "like" terms:

\[
3\sqrt{8} + 5\sqrt{2} = 3\sqrt{2 \times 2 \times 2} + 5\sqrt{2} = 3 \times 2\sqrt{2} + 5\sqrt{2} = 6\sqrt{2} + 5\sqrt{2} = 11\sqrt{2}
\]
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• **Simplify:** $\sqrt{18} - 2\sqrt{27} + 3\sqrt{3} - 6\sqrt{18}$

  I can simplify most of the radicals, and this will allow for at least a little simplification:
  \[
  \sqrt{18} - 2\sqrt{27} + 3\sqrt{3} - 6\sqrt{18} \\
  = \sqrt{9 \times 2} - 2\sqrt{9 \times 3} + 3\sqrt{3} - 6\sqrt{9 \times 2} \\
  = 2\sqrt{2} - 2 \times 3\sqrt{3} + 3\sqrt{3} - 6 \times 2\sqrt{2} \\
  = 2\sqrt{2} - 6\sqrt{3} + 3\sqrt{3} - 12\sqrt{2} \\
  = -10\sqrt{2} - 3\sqrt{3}
  \]

• **Simplify:** $2\sqrt{3} + 3\sqrt{5}$

  These two terms have "unlike" radical parts, and I can't take anything out of either radical. Then I can't simplify the expression $2\sqrt{3} + 3\sqrt{5}$ any further and my answer has to be:
  
  $2\sqrt{3} + 3\sqrt{5}$

  (expression is already fully simplified)

• **Expand:** $\sqrt{2}(3 + \sqrt{3})$

  To expand (that is, to multiply out and simplify) this expression, I first need to take the square root of two through the parentheses:
  \[
  \sqrt{2}(3 + \sqrt{3}) = \sqrt{2} \cdot 3 + \sqrt{2} \cdot \sqrt{3} \\
  = 3\sqrt{2} + \sqrt{6}
  \]

  As you can see, the simplification involved turning a product of radicals into one radical containing the value of the product (being $2 \times 3 = 6$). You should expect to need to manipulate radical products in both "directions".

• **Expand:** $\sqrt{3}(2\sqrt{3} + \sqrt{5})$

  \[
  \sqrt{3}(2\sqrt{3} + \sqrt{5}) = \sqrt{3} \cdot 2\sqrt{3} + \sqrt{3} \cdot \sqrt{5} \\
  = 2 \cdot 3 + \sqrt{15} \\
  = 6 + \sqrt{15}
  \]

• **Expand:** $(1 + \sqrt{2})(3 - \sqrt{2})$

  It will probably be simpler to do this multiplication "vertically".

  \[
  \begin{array}{c}
  3 + \sqrt{2} \\
  - \sqrt{2} \quad - \sqrt{2} \\
  \hline \\
  3 + 3\sqrt{2} - \sqrt{2} \times \sqrt{2} \\
  3 + 2\sqrt{2} - \sqrt{2} \cdot 2 = 1 + 2\sqrt{2}
  \end{array}
  \]

  Simplifying gives me:

  $3 + 2\sqrt{2} - 2 = 1 + 2\sqrt{2}$

  By doing the multiplication vertically, I could better keep track of my steps. You should use whatever multiplication method works best for you.
• Simplify \((\sqrt{3} + \sqrt{5})(\sqrt{3} - \sqrt{6})\)

I do the multiplication:

\[
\begin{align*}
\frac{\sqrt{3} + \sqrt{5}}{\sqrt{3} - \sqrt{6}} &= \frac{\sqrt{3} + \sqrt{5}}{\sqrt{3} - \sqrt{6}} \\
&= \frac{\sqrt{3}^2 - (\sqrt{5} \cdot \sqrt{6})}{\sqrt{3}^2 - (\sqrt{6})^2} \\
&= \frac{3 + \sqrt{30}}{3 - 6} \\
&= \frac{3 + \sqrt{30}}{-3} \\
&= -\frac{3}{3} - \frac{\sqrt{30}}{3} \\
&= -1 - \frac{\sqrt{30}}{3}
\end{align*}
\]

Then I complete the calculations by simplifying:

\[
\begin{align*}
\left(\sqrt{3} + \sqrt{5}\right)\left(\sqrt{3} - \sqrt{6}\right) &= \sqrt{9} + \sqrt{15} - \sqrt{18} - \sqrt{30} \\
&= 3 + \sqrt{15} - \sqrt{9} \cdot \sqrt{2} - \sqrt{30} \\
&= 3 + \sqrt{15} - 3 - \sqrt{30} \\
&= -3 - \sqrt{30}
\end{align*}
\]

• Simplify: \((\sqrt{3} + \sqrt{5})(\sqrt{3} - \sqrt{5})\)

I do the multiplication:

\[
\begin{align*}
\frac{\sqrt{3} + \sqrt{5}}{\sqrt{3} - \sqrt{5}} &= \frac{\sqrt{3} + \sqrt{5}}{\sqrt{3} - \sqrt{5}} \\
&= \frac{\sqrt{3}^2 - (\sqrt{5} \cdot \sqrt{5})}{\sqrt{3}^2 - (\sqrt{5})^2} \\
&= \frac{3 + 5}{3 - 5} \\
&= \frac{8}{-2} \\
&= -4
\end{align*}
\]

Then I simplify:

\[
\begin{align*}
\left(\sqrt{3} + \sqrt{5}\right)\left(\sqrt{3} - \sqrt{5}\right) &= \sqrt{9} + \sqrt{\cancel{15}} - \sqrt{\cancel{15}} - \sqrt{25} \\
&= \sqrt{9} - \sqrt{25} \\
&= 3 - 5 \\
&= -2
\end{align*}
\]

Note in the last example above how I ended up with all whole numbers. (Okay, technically they're integers, but the point is that the terms do not include any radicals.) I multiplied two radical "binomials" together and got an answer that contained no radicals. You may also have noticed that the two "binomials" were the same except for the sign in the middle: one had a "plus" and the other had a "minus". This pair of factors, with the second factor differing only in the one sign in the middle, is very important; in fact, this "same except for the sign in the middle" second factor has its own name:
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Given the radical expression $\sqrt{a} + \sqrt{b}$, the "conjugate" is the expression $\sqrt{a} - \sqrt{b}$.

The conjugate (KAHN-juh-ghitt) has the same numbers but the opposite sign in the middle. So not only is $\sqrt{a} - \sqrt{b}$ the conjugate of $\sqrt{a} + \sqrt{b}$, but $\sqrt{a} + \sqrt{b}$ is the conjugate of $\sqrt{a} - \sqrt{b}$.

When you multiply conjugates, you are doing something similar to what happens with a difference of squares:

When you multiply the factors $a + b$ and $a - b$, the middle "ab" terms cancel out:

$$
\begin{align*}
\frac{a + t}{a - \frac{b}{t}} &= \frac{-a\frac{b}{t} - \frac{b^2}{t}}{a^2 - \frac{b^2}{t}} \\
\frac{a^2 + ab}{\frac{a^2}{t} - \frac{b^2}{t}}
\end{align*}
$$

The same thing happens when you multiply conjugates:

$$
\begin{align*}
\frac{\sqrt{a} + \sqrt{b}}{\sqrt{a} - \sqrt{b}} &= \frac{-\sqrt{a\sqrt{b}} - \sqrt{b\sqrt{a}}}{\sqrt{a} - \sqrt{b}} \\
&= \frac{-\sqrt{ab} - \sqrt{ab}}{\sqrt{a} - \sqrt{b}}
\end{align*}
$$

We will see shortly why this matters. To get to that point, let's take a look at fractions containing radicals in their denominators.

활동 Dividing by Square Roots

Just as you can swap between the multiplication of radicals and a radical containing a multiplication, so also you can swap between the division of roots and one root containing a division.

- Simplify: $\frac{\sqrt{8}}{\sqrt{2}}$

I can simplify this by working inside, and then taking the square root:

$$
\frac{\sqrt{8}}{\sqrt{2}} = \frac{\sqrt{8}}{\sqrt{2}} = \frac{\sqrt{2} \times 2 \times \sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{2}}{\sqrt{2}} = 2
$$

...or else by splitting the division into two radicals, simplifying, and cancelling:
• Simplify: $\frac{\sqrt{3}}{\sqrt{25}}$

\[ \frac{\sqrt{3}}{\sqrt{25}} = \frac{\sqrt{3}}{\sqrt{5\times5}} = \frac{\sqrt{3}}{5} \]

\[ a^2 - b^2 = (a + b)(a - b) \]

• Simplify: $\frac{\sqrt{25}}{\sqrt{3}}$

\[ \frac{\sqrt{25}}{\sqrt{3}} = \frac{\sqrt{25\times3}}{\sqrt{3}} = \frac{5}{\sqrt{3}} \]

This looks very similar to the previous exercise, but this is the "wrong" answer. Why? Because the denominator contains a radical. The denominator must contain no radicals, or else it's "wrong". (Why "wrong" in quotes? Because this issue may matter to your instructor right now, but it probably won't later on. It's like when you were in elementary school and improper fractions were "wrong" and you had to convert everything to mixed numbers instead. But now that you're in algebra, improper fractions are fine, even preferred. Once you get to calculus or beyond, they won't be so uptight about where the radicals are.)

To get the "right" answer, I must "rationalize" the denominator. That is, I must find some way to convert the fraction into a form where the denominator has only "rational" (fractional or whole number) values. But what can I do with that radical-three? I can't take the 3 out, because I don't have a pair of threes.

Thinking back to those elementary-school fractions, you couldn't add them unless they had the same denominators. To create these "common" denominators, you would multiply, top and bottom, by whatever the denominator needed. Anything divided by itself is just 1, and multiplying by 1 doesn't change the value of whatever you're multiplying by the 1. But multiplying that "whatever" by a strategic form of 1 could make the necessary computations possible, such as:

\[ \frac{2}{5} + \frac{3}{7} = \left( \frac{2}{5} \right) \left( \frac{7}{7} \right) + \left( \frac{3}{7} \right) \left( \frac{5}{5} \right) = \frac{14}{35} + \frac{15}{35} = \frac{29}{35} \]

We can use the same technique to rationalize radical denominators.

I could take a 3 out of the denominator if I had two factors of 3 inside the radical. I can create this pair of 3's by multiplying by another copy of root-three. If I multiply top and bottom by root-three, then I will have multiplied the fraction by a strategic form of 1. I won't have changed the value, but simplification will now be possible:

\[ \frac{5}{\sqrt{3}} = \left( \frac{5}{\sqrt{3}} \right) \left( \frac{\sqrt{3}}{\sqrt{3}} \right) = \frac{5\sqrt{3}}{3} \]

This last form, "five, root-three, divided by three", is the "right" answer they're looking for.
• **Simplify:** \( \frac{6\sqrt{2}}{\sqrt{3}} \)

\[
\frac{6\sqrt{2}}{\sqrt{3}} = \left( \frac{6\sqrt{2}}{\sqrt{3}} \right) \left( \frac{\sqrt{3}}{\sqrt{3}} \right) = \frac{6\sqrt{2} \times \sqrt{3}}{\sqrt{3} \times 3} = \frac{6\sqrt{6}}{3} = \frac{2\sqrt[3]{6^2}}{\sqrt{3}} = 2\sqrt[3]{2}
\]

Don't stop once you've rationalized the denominator. As the above demonstrates, you should always check to see if something remains to be simplified.

• **Simplify:** \( \frac{3}{2+\sqrt{2}} \)

This expression is in the "wrong" form, due to the radical in the denominator. But if I try to multiply through by root-two, I won't get anything useful:

\[
\left( \frac{3}{2+\sqrt{2}} \right) \left( \frac{\sqrt{2}}{\sqrt{2}} \right) = \frac{3\sqrt{2}}{\sqrt{2}(2+\sqrt{2})}
\]

\[
= \frac{3\sqrt{2}}{2\sqrt{2} + \sqrt{2}\sqrt{2}} = \frac{3\sqrt{2}}{2\sqrt{2} + 2}
\]

Multiplying through by another copy of the whole denominator won't help, either:

\[
\left( \frac{3}{2+\sqrt{2}} \right) \left( \frac{2+\sqrt{2}}{2+\sqrt{2}} \right) = \frac{3(2+\sqrt{2})}{(2+\sqrt{2})(2+\sqrt{2})}
\]

\[
= \frac{6+3\sqrt{2}}{4+4\sqrt{2}+2} = \frac{6+3\sqrt{2}}{6+4\sqrt{2}}
\]

But look what happens when I multiply by the same numbers, but with the opposite sign in the middle:

\[
\frac{2 + \sqrt{2}}{2 - \sqrt{2}}
\]

\[
= \frac{-2\sqrt{2} - \sqrt{2}\sqrt{2}}{4 + 2\sqrt{2}}
\]

\[
= \frac{-2\sqrt{2} - \sqrt{2}\sqrt{2}}{4 + 2\sqrt{2}}
\]

\[
= \frac{4 + 2\sqrt{2}}{4 + 2\sqrt{2}} - \frac{2}{2} = 2
\]

This multiplication made the radical terms cancel out, which is exactly what I want. This "same numbers but the opposite sign in the middle" thing is the "conjugate" of the original expression. By using the conjugate, I can do the necessary rationalization.
Do not try to reach inside the numerator and rip out the 6 for "cancellation". The only thing that factors out of the numerator is a 3, but that won't cancel with the 2 in the denominator. Nothing cancels!

- Simplify: \( \frac{1+\sqrt{7}}{2-\sqrt{7}} \)

I'll multiply by the conjugate in order to "simplify" this expression. The denominator's multiplication results in a whole number (okay, a negative, but the point is that there aren't any radicals):

\[
\frac{2 - \sqrt{7}}{2 + \sqrt{7}} \times \frac{2 - \sqrt{7}}{2 - \sqrt{7}} = \frac{4 - 2\sqrt{7} - 2\sqrt{7}}{4 + 7} = \frac{-3}{9} = -\frac{1}{3}
\]

The numerator's multiplication looks like this:

\[
\frac{1 + \sqrt{7}}{2 + \sqrt{7}} \times \frac{2 + \sqrt{7}}{2 + \sqrt{7}} = \frac{9 + 3\sqrt{7}}{2 + 2\sqrt{7} + \sqrt{7} + 7} = 9 + 3\sqrt{7}
\]

Then the simplified (rationalized) form is:

\[
\frac{1+\sqrt{7}}{2-\sqrt{7}} = \left(\frac{1+\sqrt{7}}{2-\sqrt{7}}\right) \left(\frac{2+\sqrt{7}}{2+\sqrt{7}}\right) = \frac{9 + 2\sqrt{7}}{-3} = \frac{3(3 + \sqrt{7})}{-3} = -\left(3 + \sqrt{7}\right) = -3 - \sqrt{7}
\]

It can be helpful to do the multiplications separately, as shown above. Don't try to do too much at once, and make sure to check for any simplifications when you're done with the rationalization.

Operations with cube roots, fourth roots, and other higher-index roots work similarly to square roots.
Algebra 2 Semester 1

Simplifying Higher-Index Terms

• Simplify \( \sqrt[4]{16} \)

Just as I can pull from a square (or second) root anything that I have two copies of, so also I can pull from a fourth root anything I've got four of:

\[
\sqrt[4]{16} = \sqrt{2 \times 2 \times 2 \times 2} = 2
\]

If you have a cube root, you can take out any factor that occurs in threes; in a fourth root, take out any factor that occurs in fours; in a fifth root, take out any factor that occurs in fives; etc.

• Simplify the cube root: \( \sqrt[3]{8} \)

\[
\sqrt[3]{8} = \frac{2 \times 2 \times 2}{3} = 2
\]

• Simplify the cube root: \( \sqrt[3]{54} \)

\[
\sqrt[3]{54} = \frac{3 \times 3 \times 3}{3} \cdot \sqrt[3]{2} = 3 \sqrt[3]{2}
\]

• Simplify: \( \sqrt[3]{48} \)

\[
\sqrt[3]{48} = \frac{2 \times 2 \times 2 \times 2}{3} = 2 \sqrt[3]{2} = 2 \sqrt[3]{2}
\]

• Simplify: \( 4 \sqrt[3]{27} \)

\[
4 \sqrt[3]{27} = 4 \sqrt[3]{3 \times 3 \times 3} = 4 \times 3 = 12
\]

• Simplify: \( 5 \sqrt[5]{32x^{10}y^6z^7} \)

\[
5 \sqrt[5]{32x^{10}y^6z^7} = \frac{2 \times 2 \times 2 \times x^2 \times x^2 \times x^2 \times y^2 \times y^2 \times y^2 \times z^2 \times z^2 \times z^2}{5} = 2x^2 \sqrt[5]{y^2z^2}
\]

Higher Index Roots

Multiplying Higher-Index Roots

• Simplify: \( \sqrt[3]{9} \sqrt[3]{24} \)

\[
\sqrt[3]{9} \sqrt[3]{24} = \sqrt[3]{3 \times 3 \times 2 \times 2 \times 2} = 3 \times 2 = 6
\]
Algebra 2 Semester 1

- Simplify: \(\sqrt[4]{5}(2\sqrt[4]{100})\)

\[
\begin{align*}
\sqrt[4]{5}(2\sqrt[4]{100}) &= 2\sqrt[4]{5\times5\times2\times2\times5} \\
&= 2\sqrt[4]{25\times5\times5\times2\times2} \\
&= 2\times5 \frac{5}{2}\times2 = 10 \frac{5}{12}
\end{align*}
\]

Adding Higher-Index Roots

- Simplify: \(\frac{3}{\sqrt[3]{8}} + \frac{3}{\sqrt[3]{64}}\)

\[
\frac{3}{\sqrt[3]{8}} + \frac{3}{\sqrt[3]{64}} = \frac{3}{2\times2} + \frac{3}{4\times4\times4} = 2 + 4 = 6
\]

- Simplify: \(\frac{3}{\sqrt[3]{81}} + 5\sqrt[3]{3}\)

\[
\frac{3}{\sqrt[3]{81}} + 5\sqrt[3]{3} = \frac{3}{3\times3} + 5\sqrt[3]{3} = \frac{3}{3} + 5\sqrt[3]{3} = 8\sqrt[3]{3}
\]

Dividing Higher-Index Roots

1) Simplify: \(\frac{3}{\sqrt[3]{27}}\)

\[
\frac{3}{\sqrt[3]{27}} = \frac{3}{3\times3\times3} = \frac{3}{27} = \frac{3}{3} = \frac{3}{3}
\]

- Simplify: \(\frac{3}{\sqrt[5]{5}}\)

I can't simplify this expression properly, because I can't simplify the radical in the denominator down to whole numbers:

\[
\frac{3}{\sqrt[5]{5}} = \frac{3}{5} \quad \text{not simplified}
\]

To rationalize a denominator containing a square root, I needed two copies of whatever factors were inside the radical. For a cube root, I'll need three copies. So that's what I'll multiply onto this fraction:

\[
\left( \frac{3}{\sqrt[5]{5}} \right) \left( \frac{3\times3\times3}{3\times3\times3} \right) = \frac{3 \times 3 \times 25}{5 \times 5 \times 5} = \frac{3 \times 25}{5} = \frac{3 \times 25}{5}
\]
Algebra 2 Semester 1

- Simplify: \( \frac{4\sqrt{5}}{\sqrt{72}} \)

Since 72 = 8 \times 9 = 2 \times 2 \times 2 \times 3 \times 3, I won't have enough of any of the denominator's factors to get rid of the radical. To simplify a fourth root, I would need four copies of each factor. For this denominator's radical, I'll need two more 3s and one more 2:

\[
\frac{4\sqrt{5}}{\sqrt{72}} = \frac{4\sqrt{5}}{4\sqrt{2} \times 2 \times 2 \times 3 \times 3} = \frac{\sqrt{5}}{\sqrt{2} \times 2 \times 2 \times 3 \times 3} \cdot \frac{\sqrt{2} \times 2 \times 2 \times 3 \times 3}{\sqrt{2} \times 2 \times 2 \times 3 \times 3} = \frac{4\sqrt{5} \times 2 \times 2 \times 3 \times 3}{4 \times 2 \times 2 \times 3 \times 3} = \frac{4\sqrt{5}}{6}
\]

A Special Case of Rationalizing

If your class has covered the formulas for factoring the sums and differences of cubes, then you might encounter a special case of rationalizing denominators. The reasoning and methodology are similar to the "difference of squares" conjugate process for square roots.

- Simplify: \( \frac{2}{1 + \frac{3}{\sqrt[3]{4}}} \)

I would like to get rid of the cube root, but multiplying by the conjugate won't help much:

\[
\left(1 + \frac{3}{\sqrt[3]{4}}\right)\left(1 - \frac{3}{\sqrt[3]{4}}\right) = 1 - \frac{9}{\sqrt[3]{4}} = 1 - \frac{3}{2} = \frac{1}{2}
\]

But I can "create" a sum of cubes, just as using the conjugate allowed me to create a difference of squares earlier. Using the fact that \(a^3 + b^3 = (a + b)(a^2 - ab + b^2)\), and letting \(a = 1\) and \(b = \sqrt[3]{4}\), I get:

\[
\left(1 + \frac{3}{\sqrt[3]{4}}\right)\left(1 - \frac{3}{\sqrt[3]{4}} + \frac{3}{\sqrt[3]{16}}\right) = \left(1 + \frac{3}{\sqrt[3]{4}}\right)\left(1 - \frac{3}{\sqrt[3]{4}} + \frac{3}{2}\right) = 1 + \frac{3}{\sqrt[3]{4}} = 1 + 4 = 5
\]

If I multiply, top and bottom, by the second factor in the sum-of-cubes formula, then the denominator will simplify with no radicals:

\[
\frac{\frac{2}{1 + \frac{3}{\sqrt[3]{4}}}}{1 - \frac{3}{\sqrt[3]{4}} + \frac{3}{\sqrt[3]{16}}} = \frac{2\left(1 - \frac{3}{\sqrt[3]{4}} + \frac{3}{\sqrt[3]{16}}\right)}{1 + \frac{3}{\sqrt[3]{4}}\left(1 - \frac{3}{\sqrt[3]{4}} + \frac{3}{\sqrt[3]{16}}\right)} = \frac{2 - 2\frac{3}{\sqrt[3]{4}} + 4\frac{3}{\sqrt[3]{16}}}{5}
\]
Naturally, if the sign in the middle of the original denominator had been a "minus", I'd have
applied the "difference of cubes" formula to do the rationalization. This sort of "rationalize the
denominator" exercise almost never comes up. But if you see this in your homework, expect
one of these on your next test.

Radicals Expressed With Exponents

Radicals can be expressed as fractional exponents. Whatever is the index of the radical
becomes the denominator of the fractional power. For instance:

\[ \sqrt[5]{9} = \frac{\sqrt[5]{9}}{5} = 9^{\frac{1}{5}} = 3 \]

The second root became a one-half power. A cube root would be a one-third power, a fourth
root would be a one-fourth power, and so forth. This conversion process will matter a lot more
once you get to calculus. For now, it allows you to simplify some expressions that you might
otherwise not have been able to.

Express \( \frac{\sqrt[3]{2}}{\sqrt[4]{2}} \) as a single radical term.

I will convert the radicals to exponential expressions, and then apply exponent rules
to combine the factors:

\[ \frac{\sqrt[3]{2}}{\sqrt[4]{2}} = 2^{\frac{1}{3}} \cdot 2^{\frac{1}{4}} = 2^{\frac{1}{3} + \frac{1}{4}} = 2^{\frac{7}{12}} = \frac{\sqrt[12]{2^7}}{2} \]

- Simplify: \( \frac{\sqrt[5]{5}}{\sqrt[5]{5}} \)

\[ \frac{\sqrt[5]{5}}{\sqrt[5]{5}} = \frac{5^{\frac{1}{5}}}{5^{\frac{1}{5}}} = 5^{\frac{1}{5} - \frac{1}{5}} = 5^{0} = 1 \]

\[ = \frac{1}{5} = \left( \frac{1}{\sqrt[5]{5}} \right) \left( \frac{\sqrt[5]{5}}{\sqrt[5]{5}} \right) \]

\[ = \frac{\sqrt[5]{5^5}}{5} = \frac{5}{5} \]

A Few Other Considerations

Usually, we cannot have a negative inside a square root. (The exception is for "imaginary"
numbers. If you haven't done the number "i" yet, then you haven't done imaginaries.) So, for
instance, \( \sqrt{-4} \) is not possible. Do not try to say something like " \( \sqrt{-4} = -2 \)", because it's not
true: \( (-2)^2 = +4 \neq -4 \). You must have a positive inside the square root. This can be important
for defining and graphing functions.
Find the domain of the following:

\[ x = \sqrt{x - 2} \]

The fact that I have the expression \( x - 2 \) inside a square root requires that \( x - 2 \) be zero or greater, so I must have \( x - 2 \geq 0 \). Solving, I get:

**domain:** \( x \geq 2 \)

On the other hand, you CAN have a negative inside a cube root (or any other odd root). For instance:

\[ \sqrt[3]{-8} = -2 \]

...because \((-2)^3 = -8\).

Find the domain of the following:

\[ y = \sqrt[3]{x - 2} \]

For \( \sqrt[3]{x - 2} \), there is NO RESTRICTION on the value of \( x \), because \( x - 2 \) is welcome to be negative inside a cube root. Then the domain is:

**domain: all** \( x \)
(1) Add the monomials: \((9x - 6) + (-5x + 7)\)
   (a) \(14x + 1\)
   (b) \(-4x + 1\)
   (c) \(4x + 1\)
   (d) \(4x + 13\)

(2) Subtract the monomial: \((9x - 6) - (-5x + 7)\)
   (a) \(14x + 13\)
   (b) \(-4x + 1\)
   (c) \(4x + 1\)
   (d) \(4x + 13\)

(3) Multiply the monomial: \((e^2 f^4)(e^2 f^2)\)
   (a) \(ef^{10}\)
   (b) \(e^2 f^8\)
   (c) \(e^4 f^6\)
   (d) \(e^4 f^8\)

(4) Divide the monomial: \(\frac{13u^7v^7}{26u^7v}\)
   (a) \(-7uv^5\)
   (b) \(-7uv^7\)
   (c) \(\frac{v^6}{2}\)
   (d) \(\frac{uv^6}{13}\)

(5) Simplify: \(\frac{9x^0y^{-8}}{z^{-8}}\)
   (a) \(\frac{9y^8}{z^8}\)
   (b) \(\frac{9z^8}{y^8}\)
   (c) \(\frac{9x^z}{y^8}\)
   (d) \(9y^8z^8\)

(6) Add the polynomials: \((x^2 y + 3x + 2) + (2 + 2xy^2 + 3x)\)
   (a) \(x^2 + 2xy^2 + 6x + 4\)
   (b) \(3x^2 y + 6x + 4\)
   (c) \(3xy^2 + 6x + 4\)
   (d) \(3x^2 y^2 + 6x + 4\)
(7) Subtract the polynomials: \((x^2y + 3x + 2) - (2 + 2xy^2 + 3x)\)
(a) \(0\)
(b) \(-2x^2y\)
(c) \(x^2y - 2xy^2\)
(d) \(x^2y + xy^2\)

(8) Multiply: \((n - 5)(n - 1)\)
(a) \(n(n - 1) - 5(n - 1)\)
(b) \(n^2 - 5n + 5\)
(c) \(n^2 - 6n + 5\)
(d) \(n^2 + 5\)

(9) Divide: \((8x^8 - 8x^2 - 8x) ÷ (4x)\)
(a) \(2x^8 - 8x^2 - 28x\)
(b) \(2x^8 - 2x^2 - 2x\)
(c) \(8x^7 - 8x - 28\)
(d) \(2x^7 - 2x - 2\)

(10) Divide: \((x^2 + 2x - 63) ÷ (x + 9)\)
(a) \(x - 7\)
(b) \(x^2 - 7\)
(c) \(x + 7\)
(d) \(x^2 + 3x - 54\)

(11) Factor: \(10x^2 + 100x\)
(a) \(5(2x^2 + 20x)\)
(b) \(10x(x + 10)\)
(c) \(5x(2x + 20)\)
(d) \(x(10x + 100)\)

(12) Factor: \(x^2 + 4x - 77\)
(a) \((x + 4)(x - 77)\)
(b) \((x - 7)(x + 11)\)
(c) \((x + 7)(x - 11)\)
(d) \((x - 7)(x - 11)\)

(13) Graph the Function \(f(x) = x^3 + 3x^2 - 6x - 8\)
(14) Multiply the rational expression: \( \frac{c^2+7c+10}{c^2+2c-15} \cdot \frac{4c+12}{3c+15} \). Write in lowest terms.

(a) \( \frac{4(c+2)(c+3)}{3(c-3)(c+5)} \)
(b) \( \frac{4(c+2)(2+5)}{3} \)
(c) \( \frac{4(c+2)(c+3)}{3(c-3)} \)
(d) \( \frac{4(c+2)}{3(c-3)} \)

(15) Divide the rational expression: \( \frac{a^2-b^2}{2a-2b} \div \frac{8}{a+b} \). Write in lowest terms.

(a) \( \frac{1}{16} \)
(b) \( \frac{(a+b)^2}{16} \)
(c) \( \frac{(a-b)(a+b)}{16} \)
(d) \( \frac{a+b}{8} \)

(16) Which ratio is different from the others?

(a) 8 to 15
(b) 15: 8
(c) 8: 15
(d) \( \frac{8}{15} \)

(17) For what value(s) of x is \( \frac{2x+3}{5x+3} = \frac{2}{5} \)?

(a) x=1
(b) x=3
(c) no real numbers
(d) all real numbers

(18) Solve the rational equation \( \frac{3}{x-2} + \frac{7}{x} = \frac{-14}{x^2-2x} \).

(a) \{0\}
(b) \{-2\}
(c) \{0,-2\}
(d) \( \emptyset \)

(19) Identify the asymptotes of \( y = \frac{1}{8x+24} - 10 \).

(a) Vertical asymptote at x=-10. Horizontal asymptote at y=-3.
(b) Vertical asymptote at x=-3. Horizontal asymptote at y=-10.
(c) Vertical asymptote at x=-\( \frac{7}{4} \). Horizontal asymptote at y=10.
(d) Vertical asymptote at x=3. Horizontal asymptote at y=10
(20) Identify and classify any discontinuities in the function $\frac{x^2+4x+4}{x^2-4}$

(a) Removable discontinuity at $x=2$
(b) Non-removable discontinuities at $x=\pm2$
(c) Removable discontinuity at $x=-2$, non-removable discontinuity at $x=2$
(d) Removable discontinuity at $x=2$, non-removable discontinuity at $x=-2$
(e) Removable discontinuity at $x=\pm2$

(21) Find the Square root of $\frac{1}{\sqrt{25}}$

(a) $\frac{1}{625}$
(b) 5
(c) $\frac{1}{5}$
(d) 25

(22) Find the cube root of $\sqrt[3]{8}$.  

(a) 3  
(b) 4  
(c) $\pm2$  
(d) 2

(23) Use the product rule to simplify the radical $\sqrt{44}$

(a) 22  
(b) $\sqrt{44}$  
(c) $4\sqrt{11}$  
(d) $2\sqrt{11}$

(24) Simplify $\sqrt{-64x^{36}y^{18}}$

(a) $-4x^{12}y^{6}$
(b) $16x^{12}y^{6}$
(c) $4x^{12}y^{6}$
(d) $-4x^{18}y^{9}$

(25) Subtract by first simplifying each radical and the combining any like radicals.  

$5\sqrt{162} - 3\sqrt{18} - 4\sqrt{98}$

(a) $11\sqrt{2}$
(b) $8\sqrt{2}$
(c) $-11\sqrt{2}$
(d) $5\sqrt{2}$
(26) Multiply: \((\sqrt{3} + \sqrt{z})(\sqrt{3} - \sqrt{z})\)
(a) \(3 - 2\sqrt{z}\)
(b) \(3 - 2\sqrt{3z}\)
(c) \(3 - z\)
(d) \(3z\)

(27) Simplify \(\frac{5\sqrt{11}}{\sqrt{3}}\).
(a) \(\frac{\sqrt{33}}{15}\)
(b) \(\frac{5\sqrt{33}}{3}\)
(c) \(\frac{\sqrt{825}}{3}\)
(d) \(\frac{15\sqrt{3}}{11}\)