Algebra 2

AWA

AMERICAN WORLDWIDE ACADEMY

Semester 2

Common Core State Standards Initiative

COURSE DESCRIPTION

Algebra 2 Semester 2

Will meet graduation requirements for Algebra 2

Subject Area: Mathematics

Course Number: 1200330

Course Title: Algebra 2 Semester 2

Credit: 0.5

❖ Introduction

American Worldwide Academy's math course, *AWA Algebra 2*, focuses on the fundamental skills that are necessary for understanding the basics of algebra. This Study guide addresses essential standards of mathematics, such as number quadratic equations, exponential and logarithmic functions, and conic sections. *AWA Algebra 2* is full of practical, useful information geared to helping students recover credit for algebra while mastering the basics. This Study guide will be helpful to any student who has previously had difficulties with understanding algebraic concepts and skills.

There are six sections that cover core topics of algebra at the second course level. At the beginning of each section of study, you will see the objectives outlined that will help you master the standards for the section.

❖ Course Objectives

After completion of this course, students will know and be able to do the following:

Algebra Standards and Concepts

❖ Section 4: Quadratic Equations -

Students draw graphs of quadratic functions. They solve quadratic equations and solve these equations by factoring, completing the square and by using the quadratic formula. They also use graphing calculators to find approximate solutions of quadratic equations.

- ✓ Solve quadratic equations over the real numbers by factoring, and by using the quadratic formula.
- ✓ Solve quadratic equations over the real numbers by completing the square.
- ✓ Use the discriminant to determine the nature of the roots of a quadratic equation.
- ✓ Solve quadratic equations over the complex number system.
- ✓ Identify the axis of symmetry, vertex, domain, range and intercept(s) for a given parabola.
- ✓ Solve non-linear systems of equations with and without using technology.
- ✓ Use quadratic equations to solve real-world problems.
- ✓ Solve optimization problems.
- ✓ Use graphing technology to find approximate solutions of quadratic equations.

❖ Section 5: Logarithmic and Exponential Functions –

Students understand the concepts of logarithmic and exponential functions. They graph exponential functions and solve problems of growth and decay. They understand the inverse relationship between exponents and logarithms and use it to prove laws of logarithms and to solve equations. They convert logarithms between bases and simplify logarithmic expressions.

- ✓ Define exponential and logarithmic functions and determine their relationship.
- ✓ Define and use the properties of logarithms to simplify logarithmic expressions and to find their approximate values.
- ✓ Graph exponential and logarithmic functions.
- ✓ Prove laws of logarithms.
- ✓ Solve logarithmic and exponential equations.
- ✓ Use the change of base formula.
- ✓ Solve applications of exponential growth and decay.

❖ Section 6: Conic Sections –

Students write equations and draw graphs of conic sections (circle, ellipse, parabola, and hyperbola), thus relating an algebraic representation to a geometric one.

- ✓ Write the equations of conic sections in standard form and general form, in order to identify the conic section and to find its geometric properties (foci, asymptotes, eccentricity, etc.).
- ✓ Graph conic sections with and without using graphing technology.
- ✓ Solve real-world problems involving conic sections.

❖ Getting Started

You will learn much from this course that will help you in your future studies and career. In addition to reviewing and completing the study guide and textbook, your Final Examination will be evidence that you have mastered the standards for algebra. You will know the concepts and be able to do the skills that will earn you one full credit for Algebra 2.

If you are ready to begin, turn to the next page in this Study guide: the Progress Chart and Self-Test Schedule, which will serve as a guide to help you move through the course. Let's get started on earning that algebra credit—good luck!

Complete and Review:	Record Your Progress:
Solving Quadratic Equations	
Factoring	Date completed: ___/___/____
The Quadratic Formula	Date completed: ___/___/____
Completing the square	Date completed: ___/___/____
Work out all examples	Date completed: ___/___/____
The Discriminant	
Discriminant	Date completed: ___/___/____
One real solution	Date completed: ___/___/____
Two real solutions	Date completed: ___/___/____
No real solutions	Date completed: ___/___/____
Exponential Functions	
Definition	Date completed: ___/___/____
Graph and Properties	Date completed: ___/___/____
Logarithmic Functions	
Definition	Date completed: ___/___/____
Graph and Properties	Date completed: ___/___/____
Logarithmic scale	Date completed: ___/___/____
Natural logarithms	Date completed: ___/___/____
Base “a” to base “e”	Date completed: ___/___/____
Conic Sections	
Definitions	Date completed: ___/___/____
Equation form	Date completed: ___/___/____
Parabolas	Date completed: ___/___/____
Key points	Date completed: ___/___/____
General form	Date completed: ___/___/____
Graphing	Date completed: ___/___/____
Examples	Date completed: ___/___/____
Circles	Date completed: ___/___/____
Key points	Date completed: ___/___/____
General form	Date completed: ___/___/____
Graphing	Date completed: ___/___/____
Examples	Date completed: ___/___/____
Ellipses	Date completed: ___/___/____
Key points	Date completed: ___/___/____
General form	Date completed: ___/___/____
Graphing	Date completed: ___/___/____
Examples	Date completed: ___/___/____
Hyperbolas	Date completed: ___/___/____
Key points	Date completed: ___/___/____
General form	Date completed: ___/___/____
Graphing	Date completed: ___/___/____
Examples	Date completed: ___/___/____

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Content Review for Section 4: Quadratic Equations

Solving Quadratic Equations

A quadratic equation is an equation that could be written as

$$ax^2 + bx + c = 0$$

when a 0.

There are three basic methods for solving quadratic equations: factoring, using the quadratic formula, and completing the square.

Factoring

To solve a quadratic equation by factoring,

1. Put all terms on one side of the equal sign, leaving zero on the other side.
2. Factor.
3. Set each factor equal to zero.
4. Solve each of these equations.
5. Check by inserting your answer in the original equation.

Example 1

Solve $x^2 - 6x = 16$.

Following the steps, $x^2 - 6x = 16$ becomes $x^2 - 6x - 16 = 0$

Factor. $(x - 8)(x + 2) = 0$

Setting each factor to zero,

$$\begin{aligned} x - 8 &= 0 \quad \text{or} \quad & x + 2 &= 0 \\ x &= 8 & x &= -2 \end{aligned}$$

Then to check,

$$\begin{aligned} 8^2 - 6(8) &= 16 \quad \text{or} \quad & (-2)^2 - 6(-2) &= 16 \\ 64 - 48 &= 16 & 4 + 12 &= 16 \\ 16 &= 16 & 16 &= 16 \end{aligned}$$

Both values, 8 and –2, are solutions to the original equation.

Example 2

Solve $y^2 = -6y - 5.$

Setting all terms equal to zero, $y^2 + 6y + 5 = 0$

Factor. $(y + 5)(y + 1) = 0$

Setting each factor to 0,

$$\begin{aligned} y+5&=0 \quad \text{or} \quad & y+1&=0 \\ y&=-5 & y&=-1 \end{aligned}$$

To check, $y^2 = -6y - 5$

$$\begin{aligned} (-5)^2&=-6(-5)-5 \quad \text{or} \quad & (-1)^2&=-6(-1)-5 \\ 25&=30-5 & 1&=6-5 \\ 25&=25 & 1&=1 \end{aligned}$$

A quadratic with a term missing is called an **incomplete quadratic** (as long as the ax^2 term isn't missing).

Example 3

Solve $x^2 - 16 = 0.$

Factor.

$$(x+4)(x-4)=0$$

$$\begin{aligned} x+4&=0 \quad \text{or} \quad & x-4&=0 \\ 0&=-4 & x&=4 \end{aligned}$$

To check, $x^2 - 16 = 0$

$$\begin{aligned} (-4)^2-16&=0 \quad \text{or} \quad & (4)^2-16&=0 \\ 16-16&=0 & 16-16&=0 \\ 0&=0 & 0&=0 \end{aligned}$$

Example 4

Solve $x^2 + 6x = 0.$

Factor.

$$x(x+6)=0$$

$$\begin{aligned} x&=0 \quad \text{or} \quad & x+6&=0 \\ x&=0 & x&=-6 \end{aligned}$$

To check, $x^2 + 6x = 0$

$$\begin{aligned} (0)^2+6(0)&=0 \quad \text{or} \quad & (-6)^2+6(-6)&=0 \\ 0+0&=0 & 36+(-36)&=0 \\ 0&=0 & 0&=0 \end{aligned}$$

Example 5

Solve $2x^2 + 2x - 1 = x^2 + 6x - 5$.

First, simplify by putting all terms on one side and combining like terms.

$$\begin{array}{rr} 2x^2+2x-1= & x^2+6x-5 \\ -x^2-6x+5 & -x^2-6x+5 \\ \hline x^2-4x+4=0 & \end{array}$$

Now, factor.

$$\begin{aligned} (x-2)(x-2)&=0 \\ x-2&=0 \\ x&=2 \end{aligned}$$

To check, $2x^2 + 2x - 1 = x^2 + 6x - 5$

$$\begin{aligned} 2(2)^2+2(2)-1&=(2)^2+6(2)-5 \\ 8+4-1&=4+12-5 \\ 11&=11 \end{aligned}$$

The quadratic formula

Many quadratic equations cannot be solved by factoring. This is generally true when the roots, or answers, are not rational numbers. A second method of solving quadratic equations involves the use of the following formula:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

a, b, and *c* are taken from the quadratic equation written in its general form of

$$ax^2 + bx + c = 0$$

where *a* is the numeral that goes in front of x^2, *b* is the numeral that goes in front of *x*, and *c* is the numeral with no variable next to it (a.k.a., “the constant”).

When using the quadratic formula, you should be aware of three possibilities. These three possibilities are distinguished by a part of the formula called the discriminant. The *discriminant* is the value under the radical sign, $b^2 - 4ac$. A quadratic equation with real numbers as coefficients can have the following:

1. Two different real roots if the discriminant $b^2 - 4ac$ is a positive number.
2. One real root if the discriminant $b^2 - 4ac$ is equal to 0.
3. No real root if the discriminant $b^2 - 4ac$ is a negative number.

Example 6

Solve for x: $x^2 - 5x = -6$.

Setting all terms equal to 0,

$$x^2 - 5x + 6 = 0$$

Then substitute 1 (which is understood to be in front of the x^2), –5, and 6 for a, b, and c, respectively, in the quadratic formula and simplify.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(6)}}{2(1)}$$
$$x = \frac{5 \pm \sqrt{25 - 24}}{2}$$
$$x = \frac{5 \pm \sqrt{1}}{2}$$
$$x = \frac{5 \pm 1}{2}$$
$$x = \frac{5+1}{2} \text{ or } x = \frac{5-1}{2}$$
$$x = \frac{6}{2} \text{ or } x = \frac{4}{2}$$
$$x = 3 \text{ or } x = 2$$

Because the discriminant $b^2 - 4ac$ is positive, you get two different real roots.

Example produces rational roots. In Example , the quadratic formula is used to solve an equation whose roots are not rational.

Example 7

Solve for y: $y^2 = -2y + 2$.

Setting all terms equal to 0,

$$y^2 + 2y - 2 = 0$$

Then substitute 1, 2, and –2 for a, b, and c, respectively, in the quadratic formula and simplify.

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$y = \frac{-(2) \pm \sqrt{(2)^2 - 4(1)(-2)}}{2(1)}$$
$$y = \frac{-2 \pm \sqrt{4 + 8}}{2}$$
$$y = \frac{-2 \pm \sqrt{12}}{2}$$
$$y = \frac{-2 \pm \sqrt{4}\sqrt{3}}{2}$$
$$y = \frac{-2 \pm 2\sqrt{3}}{2}$$
$$y = \frac{{}^{1}\cancel{2}(-1 \pm \sqrt{3})}{\cancel{2}_{1}}$$
$$y = -1 + \sqrt{3} \text{ or } y = -1 - \sqrt{3}$$

Note that the two roots are irrational.

Example 8

Solve for x: $x^2 + 2x + 1 = 0$.

Substituting in the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-(2) \pm \sqrt{(2)^2 - 4(1)(1)}}{2(1)}$$
$$x = \frac{-2 \pm \sqrt{4-4}}{2}$$
$$x = \frac{-2 \pm \sqrt{0}}{2}$$
$$x = \frac{-2}{2} = -1$$

Since the discriminant $b^2 - 4ac$ is 0, the equation has one root.

The quadratic formula can also be used to solve quadratic equations whose roots are imaginary numbers, that is, they have no solution in the real number system.

Example 9

Solve for x: $x(x + 2) + 2 = 0$, or $x^2 + 2x + 2 = 0$.

Substituting in the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-(2) \pm \sqrt{(2)^2 - 4(1)(2)}}{2(1)}$$
$$x = \frac{-2 \pm \sqrt{4-8}}{2}$$
$$x = \frac{-2 \pm \sqrt{-4}}{2}$$

Since the discriminant $b^2 - 4ac$ is negative, this equation has no solution in the real number system.

But if you were to express the solution using imaginary numbers, the solutions would be

$$x = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i \text{ or } -1+i,\ -1-i.$$

Completing the square

A third method of solving quadratic equations that works with both real and imaginary roots is called completing the square.

1. Put the equation into the form $ax^2 + bx = -c$.
2. Make sure that $a = 1$ (if $a \neq 1$, multiply through the equation by $\frac{1}{a}$ before proceeding).
3. Using the value of b from this new equation, add $(\frac{b}{2})^2$ to both sides of the equation to form a perfect square on the left side of the equation.
4. Find the square root of both sides of the equation.
5. Solve the resulting equation.

Example 10

Solve for x: $x^2 - 6x + 5 = 0$.

Arrange in the form of

$$ax^2 + bx = -c$$
$$x^2 - 6x = -5$$

Because $a = 1$, add $(\frac{-6}{2})^2$, or 9, to both sides to complete the square.

$$x^2 - 6x + 9 = -5 + 9$$
$$x^2 - 6x + 9 = 4$$
$$(x-3)^2 = 4$$

Take the square root of both sides.

$$x - 3 = \pm 2$$

Solve.

$$\begin{array}{lcl} x - 3 = \pm 2 & & \\ x - 3 = +2 & \text{or} & x - 3 = -2 \\ \quad +3 \quad +3 & & \quad +3 \quad +3 \\ \quad x = 5 & \text{or} & \quad x = 1 \end{array}$$

Example 11 Solve for *y*: $y^2 + 2y - 4 = 0$.

Arrange in the form of

$$ay^2 + by = -c$$
$$y^2 + 2y = 4$$

Because $a = 1$, add $\left(\frac{2}{2}\right)^2$, or 1, to both sides to complete the square.

$$y^2 + 2y + 1 = 4 + 1$$
$$y^2 + 2y + 1 = 5$$
$$(y+1)^2 = 5$$

Take the square root of both sides.

$$y + 1 = \pm\sqrt{5}$$

Solve.

$$\begin{aligned} y + 1 &= \pm\sqrt{5} \\ -1 \quad & -1 \\ y &= -1 \pm \sqrt{5} \end{aligned}$$
$$y = -1 + \sqrt{5} \text{ or } y = -1 - \sqrt{5}$$

Example 12 Solve for *x*: $2x^2 + 3x + 2 = 0$.

Arrange in the form of

$$ax^2 + bx = -c$$
$$2x^2 + 3x = -2$$

Because $a \neq 1$, multiply through the equation by $\frac{1}{2}$.

$$x^2 + \frac{3}{2}x = -1$$

Add $\left[\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\right]^2$ or $\frac{9}{16}$ to both sides.

$$x^2 + \frac{3}{2}x + \frac{9}{16} = -1 + \frac{9}{16}$$
$$x^2 + \frac{3}{2}x + \frac{9}{16} = -\frac{7}{16}$$
$$\left(x + \frac{3}{4}\right)^2 = -\frac{7}{16}$$

Take the square root of both sides.

$$x + \frac{3}{4} = \pm\sqrt{\frac{-7}{16}}$$
$$x + \frac{3}{4} = \pm\frac{\sqrt{-7}}{\sqrt{16}}$$
$$x + \frac{3}{4} = \pm\frac{\sqrt{-7}}{4}$$
$$x = -\frac{3}{4} \pm \frac{\sqrt{-7}}{4}$$

The Discriminant

In the quadratic formula, the expression under the square root sign, $b^2 - 4ac$, is called the

discriminant.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The sign of the discriminant can be used to find the number of solutions of the corresponding quadratic equation,

$$ax^2 + bx + c = 0$$

If the discriminant $b^2 - 4ac$ is negative, then there are no real solutions of the equation . (You need complex numbers to deal with this case properly. These are usually taught in Algebra 2.)

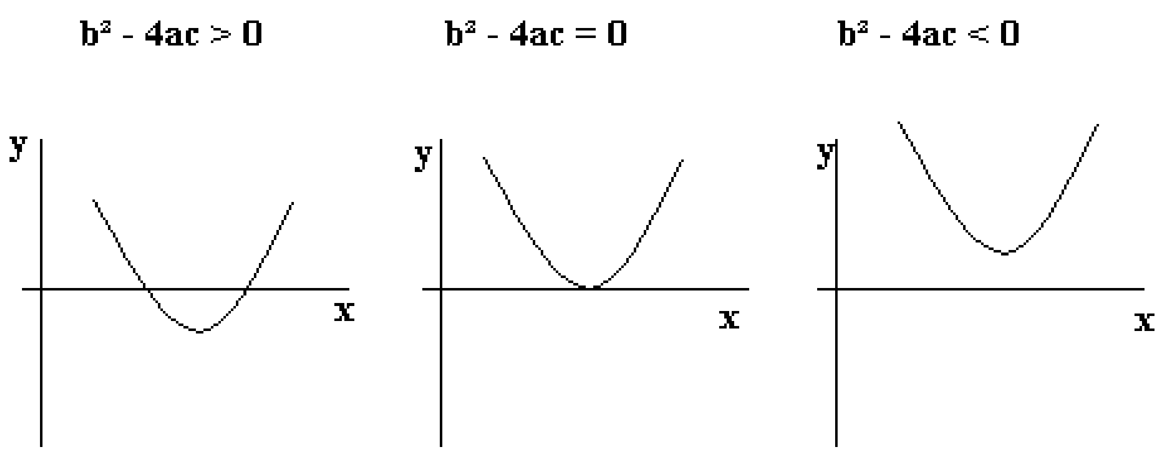
If the discriminant is zero, there is only one solution.

If the discriminant is positive, then the ± symbol means you get two answers.

The solutions of this equation correspond to the x-intercepts of the parabola

$$y = ax^2 + bx + c.$$

So, you can also use the discriminant to find the number of x-intercepts of a parabola.



Example 1:

Solve the quadratic equation.

$$x^2 - x - 12 = 0$$

Here $a = 1$, $b = -1$, and $c = -12$. Substituting, we get:

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-12)}}{2(1)}$$

Simplify.

$$x = \frac{1 \pm \sqrt{49}}{2}$$

The discriminant is positive, so we have two solutions:

$$x = \frac{8}{2} \text{ and } \frac{-6}{2}$$

$x = 4$ and $x = -3$

In this example, the discriminant was 49, a perfect square, so we ended up with rational answers. Often, when using the quadratic formula, you end up with answers which still contain radicals.

Example 2:

Solve the quadratic equation

$$3x^2 + x - 5 = 0$$

Here, $a = 3$, $b = 1$, $c = -5$. Substituting, we get:

$$x = \frac{-1 \pm \sqrt{1^2 - (4)(3)(-5)}}{2(3)}$$

Simplify:

$$x = \frac{-1 \pm \sqrt{61}}{6}$$

This means we have two roots:

$$\frac{-1 + \sqrt{61}}{6} \text{ and } \frac{-1 - \sqrt{61}}{6}$$

Example 3:

Solve the quadratic equation.

$3x^2 + 2x + 1 = 0$

Here $a = 3$, $b = 2$, and $c = 1$. Substituting, we get:

$$x = \frac{-2 \pm \sqrt{2^2 - 4(3)(1)}}{2(3)}$$

Simplify.

$$x = \frac{-2 \pm \sqrt{-8}}{6}$$

The discriminant is negative, so this equation has no real solutions.

Content Review for Section 5: Logarithmic and Exponential Functions

Exponential functions

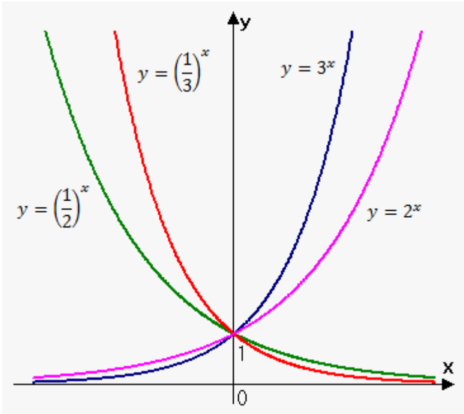
Definition

Take a > 0 and not equal to 1 . Then, the function defined by

f : **R** -> **R** : x -> a^x
is called an exponential function with base a.

Graph and properties

Let f(x) = an exponential function with a > 1.
Let g(x) = an exponential function with 0 < a < 1.



From the graphs we see that

- The domain is **R**
- The range is the set of strictly positive real numbers
- The function is continuous in its domain
- The function is increasing if a > 1 and decreasing if 0 < a < 1
- The x-axis is a horizontal asymptote

Examples:

$y = 3^x$; $y = 0.5^x$; $y = 10^{0.2x-1}$

Logarithmic functions

Take a > 0 and not equal to 1. Since the exponential functions
f : **R** -> **R** : x -> a^x
are either increasing or decreasing, the inverse functions are defined. The inverse function is called the logarithmic function with base a. We write

$\log_a(x)$

$\log_{10}(x)$ is written as log(x)

So,

$$\log_a(x) = y \iff a^y = x$$

From this we see that the domain of the logarithmic function is the set of strictly positive real numbers, and the range is **R**.

Examples: $\log_2(8) = 3$; $\log_3(\text{sqrt}(3)) = 0.5$; $\log(0.01) = -2$

From the definition it follows immediately that

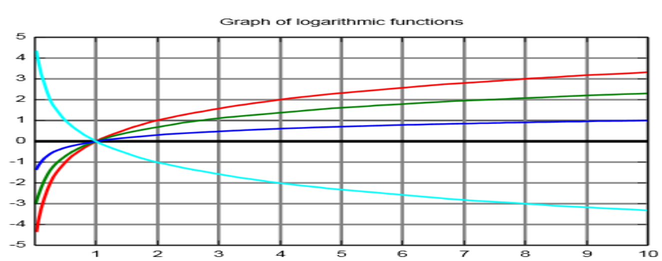
for x > 0 we have $a^{\log_a(x)} = x$

and

for all x we have $\log_a(a^x) = x$

Example: $\log(10^{2x+1}) = 2x+1$

Graph Let f(x) = a logarithmic function with a > 1.
Let g(x) = a logarithmic function with 0 < a < 1.



From the graphs we see that

- The range is **R**
- The domain is the set of strictly positive real numbers
- The function is continuous in its domain
- The function is increasing if a > 1 and decreasing if 0 < a < 1
- The y-axis is a vertical asymptote

Examples: $\log_2(x)$; $\log(2x+4)$; $\log_{0.5}(x)$

Properties

In the next 3 properties, all logarithmic functions have base a > 0. For convenience, I don't write this base a.

$$\mathbf{\log(x.y) = \log(x) + \log(y)}$$

Proof : Let $\log(x.y) = u$ then $a^u = x.y$ (1)

Let $\log(x) = v$ then $a^v = x$ (2)

Let $\log(y) = w$ then $a^w = y$ (3)

From (1), (2) and (3)

$$a^u = a^v \, . \, a^w$$
$$\Rightarrow \quad a^u = a^{v+w}$$
$$\Rightarrow \quad u = v + w$$

So,

$$\log_a(x \cdot y) = \log_a(x) + \log_a(y)$$

In the same way you can prove that

$$\log_a(x/y) = \log_a(x) - \log_a(y)$$

For each real number r we have:

$$\log_a(x^r) = r \cdot \log_a(x)$$

Examples:

$$\log(x^2 y^3) = 2\log(x) + 3\log(y)$$
$$\log(x^2 / y^3) = 2\log(x) - 3\log(y)$$
$$\log(x^y) = y\log(x)$$
$$\log(2x) + \log(3x) = \log(6x^2)$$
$$3\log(x) + a\log(x) = (3+a)\log(x) = \log(x^{3+a})$$
$$0.5\log(x) = \log(\mathrm{sqrt}(x))$$

Show that $x^{\log(y)} / y^{\log(x)} = 1$

The statement implies that x and y are positive, so we can write

$$\begin{aligned} & x^{\log(y)} / y^{\log(x)} = 1 \\ \iff\ & x^{\log(y)} = y^{\log(x)} \\ \iff\ & \log(x^{\log(y)}) = \log(y^{\log(x)}) \\ \iff\ & \log(y)\cdot\log(x) = \log(x)\cdot\log(y) \end{aligned}$$

Change the base of a logarithmic function

Sometimes it is very useful to change the base of a logarithmic function.
Theorem: for each strictly positive real number a and b, different from 1, we have

$$\log_a(x) = \left(\frac{}{\log_b(a)}\right) .\ \log_b(x)$$

Proof:
We'll prove that

$$\log_b(a) . \log_a(x) = \log_b(x)$$

Let $\log_b(a) = u$ then $b^u = a$ (1)
Let $\log_a(x) = v$ then $a^v = x$ (2)
Let $\log_b(x) = w$ then $b^w = x$ (3)
From (2) and (3) we have

$$a^v = b^w$$

Using (1)

$$b^{u.v} = b^w$$

So,

$$u\cdot v = w$$

$$\Rightarrow \log_b(a) \cdot \log_a(x) = \log_b(x)$$

Calculating images of a logarithmic function with a calculator

Examples:
- $\log(12.5) = 1.0969$
- $\log_2(12) = \log(12)/\log(2) = 3.58496$
- $\log(1/154) = -\log(154) = -2.1875$
- $\log_7(0.5^{14}) = 14 \log(0.5)/\log(7) = -4.9869$
- $\log(-12.4)$ is not defined

Logarithmic scale

- Take a function with an equation of the form $y = 10^{ax+b}$. If you plot this function with a logarithmic scale on the y-axis, then the graph is a straight line with slope a.
- Take a function with an equation of the form $y = \log(a \cdot x^p)$. If you plot this function with a logarithmic scale on the x-axis, then the graph is a straight line with slope p.
- Take a function with an equation of the form $y = a \cdot x^p$. If you plot this function with a logarithmic scale on the x-axis and on the y-axis, then the graph is a straight line with slope p.

Common Logarithms: Base 10

Sometimes you will see a logarithm written **without** a base, like this:

$$\log(100)$$

This usually means that the base is really 10.

It is how many times you need to use 10 in a multiplication, to get the desired number.

Example: $\mathbf{\log(1000) = \log_{10}(1000) = 3}$

Natural Logarithms: Base "e"

Another base that is often used is e (Euler's Number) which is approximately 2.71828.

It is how many times you need to use "e" in a multiplication, to get the desired number.

Example: $\mathbf{\ln(7.389) = \log_e(7.389) \approx 2}$

Because $\mathbf{2.71828^2 \approx 7.389}$

But Sometimes There Is Confusion ... !

Mathematicians use "log" (instead of "ln") to mean the natural logarithm. This can lead to confusion:

Example	Engineer Thinks	Mathematician Thinks	
$\log(50)$	$\log_{10}(50)$	$\log_e(50)$	confusion
$\ln(50)$	$\log_e(50)$	$\log_e(50)$	no confusion
$\log_{10}(50)$	$\log_{10}(50)$	$\log_{10}(50)$	no confusion

So, be careful when you read "log" that you know what base they mean!

Transformation of an exponential function with base a, to an exponential function with base e

In applied mathematics, the exponential functions with a base different from e are underused. This is because an exponential function with base a can be converted to an exponential function with base e.
We now show the procedure.

NOTE: The number "a" is a strictly positive number.

$$a^k = e^r$$

$\leftrightarrow$ $\ln(a^k) = r$

$\leftrightarrow$ $r = k.\ln(a)$

Then

$$(a^k)^x = (e^r)^x$$

$\leftrightarrow$ $r = k.\ln(a)$ for all x

$\leftrightarrow$ $a^{kx} = e^{rx}$

$\leftrightarrow$ $r = k\cdot\ln(a)$ for all x

The functions $A\cdot a^{kx}$ and $A\cdot e^{rx}$ are identical if and only if $r = k\cdot\ln(a)$

Example:

$14 \cdot 3^{2.7x}$ is identical with $14 \cdot e^{2.97x}$

$6 \cdot (0.25)^{-x}$ is identical with $6 \cdot e^{1.39x}$

Working with the functions with base e, has many benefits for algebraic calculations.

The product of the functions with base e, from the previous example, is much easier to work with.

Compare the expressions:

$$14\ e^{2.97\ x}\ 6\ e^{1.39\ x}$$

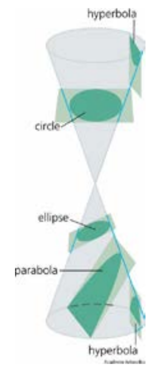
and

$$14\ 3^{2.7x}\ 6\ (0.25)^{-x}.$$

Content Review for Section 6: Conic Sections

Conic Sections: An Overview

Conic sections are the curves which can be derived from taking slices of a "double-napped" cone. (A double-napped cone, in regular English, is two cones "nose to nose", with the one cone balanced perfectly on the other.) "Section" here is used in a sense similar to that in medicine or science, where a sample (from a biopsy, for instance) is frozen or suffused with a hardening resin, and then extremely thin slices ("sections") are shaved off for viewing under a microscope. If you think of the double-napped cones as being hollow, the curves we refer to as conic sections are what results when you section the cones at various angles.



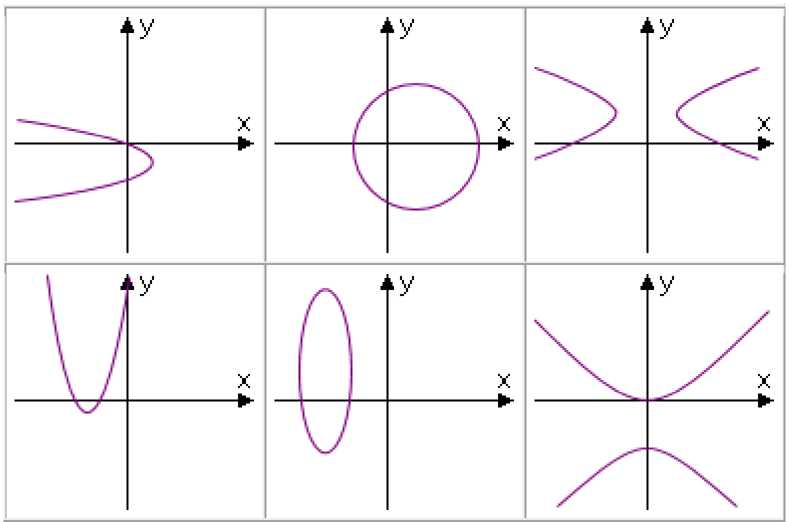
This lesson will instead concentrate on: finding curves, given points and other details; finding points and other details, given curves; and setting up and solving conics equations to solve typical word problems.

There are some basic terms that you should know for this topic:

- **center**: the point (h, k) at the center of a circle, an ellipse, or an hyperbola.
- **vertex**: in the case of a parabola, the point (h, k) at the "end" of a parabola; in the case of an ellipse, an end of the major axis; in the case of an hyperbola, the turning point of a branch of an hyperbola; the plural form is "vertices".
- **focus**: a point from which distances are measured in forming a conic; a point at which these distance-lines converge, or "focus".
- **directrix**: a line from which distances are measured in forming a conic.
- **axis**: a line perpendicular to the directrix passing through the vertex of a parabola; also called the "axis of symmetry".
- **major axis:** a line segment perpendicular to the directrix of an ellipse and passing through the foci; the line segment terminates on the ellipse at either end; also called the "principal axis of symmetry"; the half of the major axis between the center and the vertex is the semi-major axis.
- **minor axis:** a line segment perpendicular to and bisecting the major axis of an ellipse; the segment terminates on the ellipse at either end; the half of the minor axis between the center and the ellipse is the semi-minor axis.
- **locus**: a set of points satisfying some condition or set of conditions; each of the conics is a locus of points that obeys some sort of rule or rules.

You may encounter additional terms, depending on your textbook. Just make sure that you understand the particular terms that come up in your homework, so you're prepared for the test.

One very basic question that comes up pretty frequently is "Given an equation, how do I know which sort of conic it is?" Just as each conic has a typical shape:



...so also each conic has a "typical" equation form, sometimes along the lines of the following:

parabola: $Ax^2 + Dx + Ey^2 = 0$
circle: $x^2 + y^2 + Dx + Ey + F = 0$
ellipse: $Ax^2 + Cy^2 + Dx + Ey + F = 0$
hyperbola: $Ax^2 - Cy^2 + Dx + Ey + F = 0$

These equations can be rearranged in various ways, and each conic has its own special form that you'll need to learn to recognize, but some characteristics of the equations above remain unchanged for each type of conic. If you keep these consistent characteristics in mind, then you can run through a quick check-list to determine what sort of conic is represented by a given quadratic equation.

Given a general-form conic equation in the form $Ax^2 + Cy^2 + Dx + Ey + F = 0$, or after rearranging to put the equation in this form (that is, after moving all the terms to one side of the "equals" sign), this is the sequence of tests you should keep in mind:

- **Are both variables squared?**

  No: It's a parabola.
  Yes: Go to the next test....

  - **Do the squared terms have opposite signs?**

    Yes: It's an hyperbola.
    No: Go to the next test....

    - **Are the squared terms multiplied by the same number?**

      Yes: It's a circle.
      No: It's an ellipse.

- **Classify the following equations according to the type of conic each represents:**

  A) $3x^2 + 3y^2 - 6x + 9y - 14 = 0$
  B) $6x^2 + 12x - y + 15 = 0$
  C) $x^2 + 2y^2 + 4x + 2y - 27 = 0$
  D) $x^2 - y^2 + 3x - 2y - 43 = 0$

  A) Both variables are squared, and both squared terms are multiplied by the same number, so this is **a circle**.

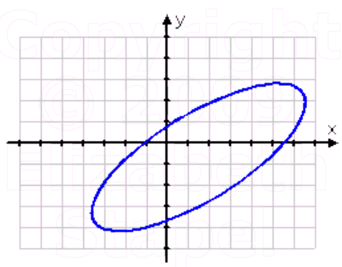
  B) Only one of the variables is squared, so this is **a parabola.**

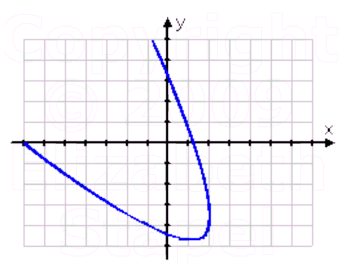
  C) Both variables are squared and have the same sign, but they aren't multiplied by the same number, so this is **an ellipse**.

  D) Both variables are squared, and the squared terms have opposite signs, so this is **an hyperbola**.

If they give you an equation with variables on either side of the "equals" sign, rearrange the terms (on paper or in your head) to get the squared stuff together on one side. Then compare with the flow-chart above to find the type of equation you're looking at.

You may have noticed, in the table of "typical" shapes (above), that the graphs either paralleled the *x*-axis or the *y*-axis, and you may have wondered whether conics can ever be "slanted", such as:





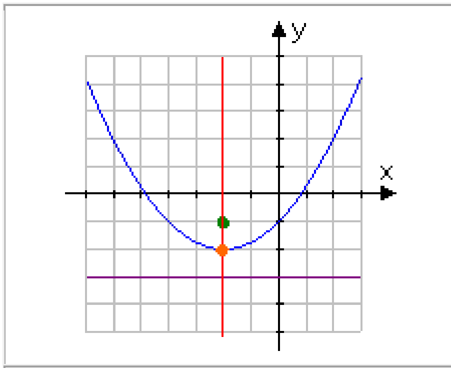
Yes, conic graphs can be "slanty", as shown above. But the equations for the "slanty" conics get so much more messy that you can't deal with them until after trigonometry. If you wondered why the coefficients in the "general conic" equations, such as $Ax^2 + Cy^2 + Dx + Ey + F = 0$, skipped the letter *B*, it's because the *B* is the coefficient of the "*xy*" term that you can't handle until after you have some trigonometry under your belt. You'll probably *never* have to deal with the "slanty" conics until calculus, when you may have to do "rotation of axes". Don't be in a rush. It's not a pretty topic.

Once you have classified a conic, what can you do with it? The following lessons give some examples

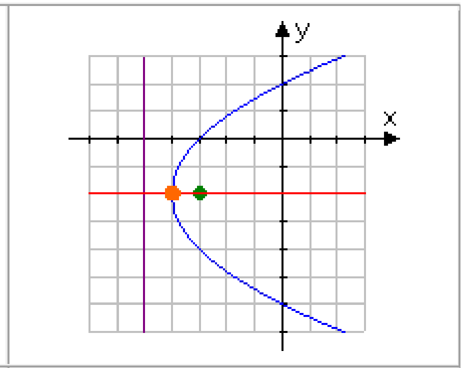
Parabolas:

In algebra, dealing with parabolas usually means graphing quadratics or finding the max/min points (that is, the vertices) of parabolas for quadratic word problems. In the context of conics, however, there are some additional considerations.

To form a parabola according to ancient Greek definitions, you would start with a line and a point off to one side. The line is called the "directrix"; the point is called the "focus". The parabola is the **curve** formed from all the points (x, y) that are equidistant from the **directrix** and the **focus**. The line perpendicular to the directrix and passing through the focus (that is, the line that splits the parabola up the middle) is called the "**axis of symmetry**". The point on this axis which is exactly midway between the focus and the directrix is the "**vertex**"; the vertex is the point where the parabola changes direction.



"regular", or vertical, parabola (in blue), with the focus (in green) "inside" the parabola, the directrix (in purple) below the graph, the axis of symmetry (in red) passing through the focus and perpendicular to the directrix, and the vertex (in orange) on the graph



"sideways", or horizontal, parabola (in blue), with the focus (in green) "inside" the parabola, the directrix (in purple) to the left of the graph, the axis of symmetry (in red) passing through the focus and perpendicular to the directrix, and the vertex (in orange) on the graph

In previous contexts, your parabolas have either been "right side up" or "upside down" graph, depending on whether the leading coefficient was positive or negative, respectively. In the context of conics, however, you will also be working with "sideways" parabolas, parabolas whose axes of symmetry parallel the *x*-axis and which open to the right or to the left.

A basic property of parabolas "in real life" is that any light or sound ray entering the parabola parallel to the axis of symmetry and hitting the inner surface of the parabolic "bowl" will be reflected back to the focus. "Parabolic dishes", such as "bionic ears" and radio telescopes, take advantage of this property to concentrate a signal onto a receiver. The focus of a parabola is always inside the parabola; the vertex is always on the parabola; the directrix is always outside the parabola.

The "general" form of a parabola's equation is the one you're used to, $y = ax^2 + bx + c$ — unless the quadratic is "sideways", in which case the equation will look something like $x = ay^2 + by + c$. The important difference in the two equations is in which variable is squared: for regular (vertical) parabolas, the *x* part is squared; for sideways (horizontal) parabolas, the *y* part is squared.

The "vertex" form of a parabola with its vertex at (h, k) is:

$$\text{regular: } y = a(x - h)^2 + k$$
$$\text{sideways: } x = a(y - k)^2 + h$$

The conics form of the parabola equation (the one you'll find in advanced or older texts) is:

$$\text{regular: } 4p(y - k) = (x - h)^2$$
$$\text{sideways: } 4p(x - h) = (y - k)^2$$

Why "(h, k)" for the vertex? Why "p" instead of "a" in the old-time conics formula? Dunno. The important thing to notice, though, is that the *h* always stays with the *x*, that the *k* always stays with the *y*, and that the *p* is always on the *unsquared* variable part. The relationship between the "vertex" form of the equation and the "conics" form of the equation is nothing more than a rearrangement:

$$y = a(x - h)^2 + k$$
$$y - k = a(x - h)^2$$
$$(1/a)(y - k) = (x - h)^2$$
$$4p(y - k) = (x - h)^2$$

In other words, the value of $4p$ is actually the same as the value of $1/a$; they're just two ways of saying the exact same thing. But this new variable *p* is one you'll need to be able to work with when you're doing parabolas in the context of conics: it represents the distance between the vertex and the focus, and also the same (that is, equal) distance between the vertex and the directrix. And $2p$ is then clearly the distance between the focus and the directrix.

- **State the vertex and focus of the parabola having the equation $(y - 3)^2 = 8(x - 5)$.**

  Comparing this equation with the conics form, and remembering that the *h* always goes with the *x* and the *k* always goes with the *y*, I can see that the center is at $(h, k) = (5, 3)$. The coefficient of the *unsquared* part is $4p$; in this case, that gives me $4p = 8$, so $p = 2$. Since the *y* part is squared and *p* is positive, then this is a sideways parabola that opens to the right. The focus is inside the parabola, so it has to be two units to the right of the vertex:

  vertex: (5, 3); focus: (7, 3)

- **State the vertex and directrix of the parabola having the equation $(x + 3)^2 = –20(y – 1)$.**

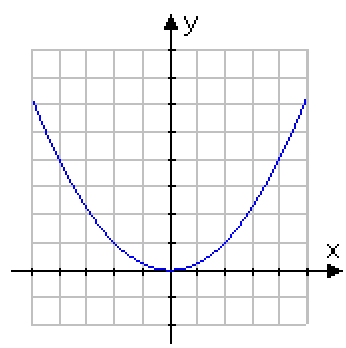
  The temptation is to say that the vertex is at (3, 1), but that would be wrong. The conics form of the equation has *subtraction* inside the parentheses, so the $(x + 3)^2$ is really $(x – (–3))^2$, and the vertex is at (–3, 1). The coefficient of the *unsquared* part is –20, and this is also the value of $4p$, so $p = –5$. Since the x part is squared and p is negative, then this is a regular parabola that opens downward. This means that the directrix, being on the outside of the parabola, is five units above the vertex.

  vertex: (–3, 1); directrix: $y = 6$

- **Graph $x^2 = 4y$ and state the vertex, focus, axis of symmetry, and directrix.**

  This is the same graphing that I've done in the past: $y = (1/4)x^2$. So I'll <u>do the graph</u> as usual:

x	$y = \frac{1}{4}x^2$
-4	$\frac{1}{4}(-4)^2 = \frac{1}{4}(16) = 4$
-2	$\frac{1}{4}(-2)^2 = \frac{1}{4}(4) = 1$
0	$\frac{1}{4}(0)^2 = \frac{1}{4}(0) = 0$
2	$\frac{1}{4}(2)^2 = \frac{1}{4}(4) = 1$
4	$\frac{1}{4}(4)^2 = \frac{1}{4}(16) = 4$



The vertex is obviously at the origin, but I need to "show" this "algebraically" by rearranging the given equation into the conics form:

$$x^2 = 4y$$
$$(x – 0)^2 = 4(y – 0)$$

This rearrangement "shows" that the vertex is at $(h, k) = (0, 0)$. The axis of symmetry is the vertical line right through the vertex: $x = 0$. (I can always check my graph, if I'm not sure about this.) The focus is "p" units from the vertex. Since the focus is "inside" the parabola and since this is a "right side up" graph, the focus has to be above the vertex.

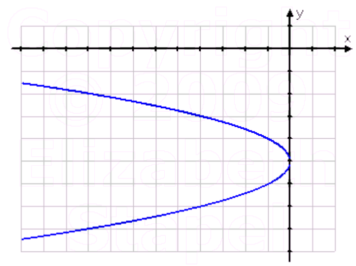
From the conics form of the equation, shown above, I look at what's multiplied on the *unsquared* part and see that $4p = 4$, so $p = 1$. Then the focus is one unit above the vertex, at (0, 1), and the directrix is the horizontal line $y = –1$, one unit below the vertex.

vertex: (0, 0); focus: (0, 1); axis of symmetry: $x = 0$; directrix: $y = –1$

- **Graph $y^2 + 10y + x + 25 = 0$, and state the vertex, focus, axis of symmetry, and directrix.**

  Since the y is squared in this equation, rather than the x, then this is a "sideways" parabola. To graph, I'll do my T-chart backwards, picking y-values first and then finding the corresponding x-values for $x = -y^2 - 10y - 25$:

y	$x=-y^2-10y-25$	(x,y)
-7	$-(-7)^2-10(-7)-25=-49+70-25=-4$	$(-4,-7)$
-6	$-(-6)^2-10(-6)-25=-36+60-25=-1$	$(-1,-6)$
-5	$-(-5)^2-10(-5)-25=-25+50-25=0$	$(0,-5)$
-4	$-(-4)^2-10(-4)-25=-16+40-25=-1$	$(-1,-4)$
-3	$-(-3)^2-10(-3)-25=-9+30-25=-4$	$(-4,-3)$



  To convert the equation into conics form and find the exact vertex, etc, I'll need to convert the equation to perfect-square form. In this case, the squared side is already a perfect square, so:

$$y^2 + 10y + 25 = -x$$
$$(y + 5)^2 = -1(x - 0)$$

  This tells me that $4p = -1$, so $p = -1/4$. Since the parabola opens to the left, then the focus is 1/4 units to the left of the vertex. I can see from the equation above that the vertex is at $(h, k) = (0, -5)$, so then the focus must be at $(-1/4, -5)$. The parabola is sideways, so the axis of symmetry is, too. The directrix, being perpendicular to the axis of symmetry, is then vertical, and is 1/4 units to the right of the vertex. Putting this all together, I get:

  vertex: $(0, -5)$; focus: $(-1/4, -5)$; axis of symmetry: $y = -5$; directrix: $x = 1/4$

- **Find the vertex and focus of $y^2 + 6y + 12x - 15 = 0$**

  The y part is squared, so this is a sideways parabola. I'll get the y stuff by itself on one side of the equation, and then complete the square to convert this to conics form.

$$y^2 + 6y - 15 = -12x$$
$$y^2 + 6y + 9 - 15 = -12x + 9$$
$$(y + 3)^2 - 15 = -12x + 9$$
$$(y + 3)^2 = -12x + 9 + 15 = -12x + 24$$
$$(y + 3)^2 = -12(x - 2)$$
$$(y - (-3))^2 = 4(-3)(x - 2)$$

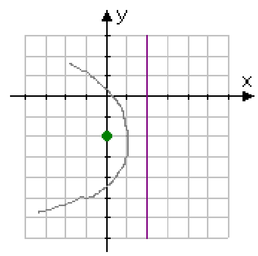
  Then the vertex is at $(h, k) = (2, -3)$ and the value of p is -3. Since y is squared and p is negative, then this is a sideways parabola that opens to the left. This puts the focus 3 units to the left of the vertex.

  vertex: $(2, -3)$; focus: $(-1, -3)$

You will also need to work the other way, going from the properties of the parabola to its equation.

- **Write an equation for the parabola with focus at (0, –2) and directrix $x = 2$.**

  The vertex is always halfway between the focus and the directrix, and the parabola always curves away from the directrix, so I'll do a quick graph showing the focus, the directrix, and a rough idea of where the parabola will go:



  So the vertex, exactly between the focus and directrix, must be at $(h, k) = (1, -2)$. The absolute value of p is the distance between the vertex and the focus and the distance between the vertex and the directrix. (The sign on p tells me which way the parabola faces.) Since the focus and directrix are two units apart, then this distance has to be one unit, so $|\ p\ | = 1$.

  Since the focus is to the left of the vertex and directrix, then the parabola faces left (as I'd shown in my picture) and I get a negative value for p: $p = -1$. Since this is a "sideway" parabola, then the y part gets squared, rather than the x part. So the conics form of the equation must be:

$$(y - (-2))^2 = 4(-1)(x - 1), \text{ or } \mathbf{(y + 2)^2 = -4(x - 1)}$$

- **Write an equation of the parabola with vertex (3, 1) and focus (3, 5).**

  Since the x-coordinates of the vertex and focus are the same, they are one of top of the other, so this is a regular vertical parabola, where the x part is squared. Since the vertex is below the focus, this is a right-side up parabola and p is positive. Since the vertex and focus are $5 - 1 = 4$ units apart, then $p = 4$.

  And that's all I need for my equation, since they already gave me the vertex.

$$(x - h)^2 = 4p(y - k)$$
$$(x - 3)^2 = 4(4)(y - 1)$$
$$\mathbf{(x - 3)^2 = 16(y - 1)}$$

- **Write an equation for the parabola with vertex (5, –2) and directrix $y = -5$.**

  The directrix is an horizontal line; since this line is perpendicular to the axis of symmetry, then this must be a regular parabola, where the x part is squared.

  The distance between the vertex and the directrix is $|-5 - (-2)| = |-5 + 2| = 3$. Since the directrix is below the vertex, then this is a right-side up parabola, so p is positive: $p = 3$. And that's all I need to find my equation:

  $$(x - h)^2 = 4p(y - k)$$
  $$(x - 5)^2 = 4(3)(y - (-2))$$
  $$\mathbf{(x - 5)^2 = 12(y + 2)}$$

- **An arch in a memorial park, having a parabolic shape, has a height of 25 feet and a base width of 30 feet. Find an equation which models this shape, using the *x*-axis to represent the ground. State the focus and directrix.**

  For simplicity, I'll center the curve for the arch on the y-axis, so the vertex will be at $(h, k) = (0, 25)$. Since the width is thirty, then the x-intercepts must be at $x = -15$ and $x = +15$. Obviously, this is a regular (vertical) but upside-down parabola, so the x part is squared and I'll have a negative leading coefficient.

  <u>Working backwards</u> from the x-intercepts, the equation has to be of the form $y = a(x - 15)(x + 15)$. Plugging in the known vertex value, I get:

  $$25 = a(0 - 15)(0 + 15) = -225a$$

  Then $a = -1/9$. With a being the leading coefficient from the regular quadratic equation $y = ax^2 + bx + c$, I also know that the value of $1/a$ is the same as the value of $4p$, so $1/(-1/9) = -9 = 4p$, and thus $p = -9/4$.

  The focus is 9/4 units below the vertex; the directrix is the horizontal line 9/4 units above the vertex:

  $$4p(y - h) = (x - k)^2$$
  $$4(-9/4)(y - 25) = (x - 0)^2$$
  $$\mathbf{-9(y - 25) = x^2}$$
  focus: (0, 91/4), directrix: $y = 109/4$

You could also work directly from the conics form of the parabola equation, plugging in the vertex and an x-intercept, to find the value of p:

$$4p(y - 25) = (x - 0)^2$$
$$4p(0 - 25) = (15 - 0)^2$$
$$4p(-25) = 225$$
$$4p = -225/25 = -9$$
$$p = -9/4$$

You may encounter an exercise of this sort regarding the Gateway Arch in Saint Louis, Missouri. In fact, that Arch is in the shape of an "inverted catenary" curve; in particular, a hyperbolic cosine curve. But its shape is close enough to that of a parabola for the purposes of the exercise. (If you ever visit Saint Louis, you should definitely try to visit the Arch. You can watch a movie down in the basement describing the design and construction of the Arch, and then you ride the tram up to the top of the Arch. The view is fabulous!)

1) **A radio telescope has a parabolic dish with a diameter of 100 meters. The collected radio signals are reflected to one collection point, called the "focal" point, being the focus of the parabola. If the focal length is 45 meters, find the depth of the dish, rounded to one decimal place.**

- To simplify my computations, I'll put the vertex of my parabola (that is, the base of the dish) at the origin, so $(h, k) = (0, 0)$. Since the focal length is 45, then $p = 45$ and the equation is:

$$4py = x^2$$
$$4(45)y = x^2$$
$$180y = x^2$$

This parabola extends forever in either direction, but I only care about the part of the curve that models the dish. Since the dish has a diameter of a hundred meters, then I only care about the part of the curve from $x = -50$ to $x = +50$.

The height of the edge of the dish (and thus the depth of the dish) will be the y-value of the equation at the "ends" of the modelling curve. The height of the parabola will be the same at either x-value, since they're each the same distance from the vertex, so it doesn't matter which value I use. I prefer positive values, so I'll plug $x = 50$ into my modelling equation:

$$180y = (50)^2$$
$$180y = 2500$$
$$y = 250/18$$

...or **about 13.9 meters.**

Circles

A circle is a geometrical shape, and is not of much use in algebra, since the equation of a circle isn't a function. But you may need to work with circle equations in your algebra classes.

In "primative" terms, a circle is the shape formed in the sand by driving a stick (the "center") into the sand, putting a loop of string around the center, pulling that loop taut with another stick, and dragging that second stick through the sand at the further extent of the loop of string. The resulting figure drawn in the sand is a circle.

In algebraic terms, a circle is the set (or "locus") of points (x, y) at some fixed distance r from some fixed point (h, k). The value of r is called the "radius" of the circle, and the point (h, k) is called the "center" of the circle.

The "general" equation of a circle is:

$$x^2 + y^2 + Dx + Ey + F = 0$$

The "center-radius" form of the equation is:

$$(x - h)^2 + (y - k)^2 = r^2$$

...where the h and the k come from the center point (h, k) and the r^2 comes from the radius value r. If the center is at the origin, so

$(h, k) = (0, 0)$, then the equation simplifies to $x^2 + y^2 = r^2$.

Note: The "general" form may be given in your book using different letters for the coefficients, but the "center-radius" form will use the same letters as shown above: The center is always denoted by "(h, k)" and the radius is always denoted by "r". Your book might also refer to something called the "standard" form of the circle equaion. Unfortunately, "standard form" has no fixed meaning that I've been able to determine, so you'll have to keep track of what your book intends by that term.
You can convert the "center-radius" form of the circle equation into the "general" form by multiplying things out and simplifying; you can convert in the other direction by completing the square.

The center-radius form of the circle equation comes directly from the Distance Formula and the definition of a circle. If the center of a circle is the point (h, k) and the radius is length r, then every point (x, y) on the circle is distance r from the point (h, k). Plugging this information into the Distance Formula (using r for the distance between the points and the center), we get:

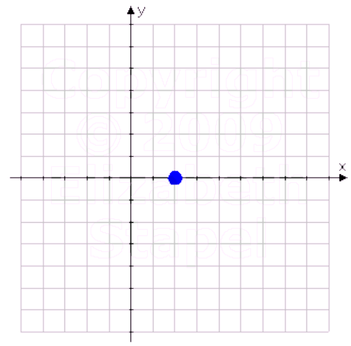
$$r = \sqrt{(x-h)^2 + (y-k)^2}$$
$$(r)^2 = \left(\sqrt{(x-h)^2 + (y-k)^2}\right)^2$$
$$r^2 = (x-h)^2 + (y-k)^2$$

You should recognize this formula; you will be expected to be able to read information from it.

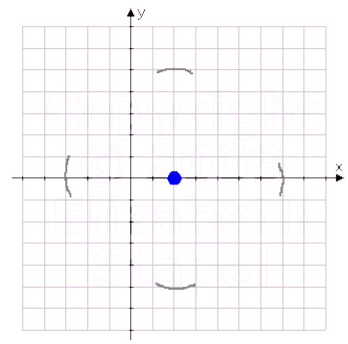
- **State the center and radius of the circle with the equation $(x-2)^2 + y^2 = 5^2$, and sketch the circle.**

  The y^2 term means the same thing as $(y-0)^2$, so the equation is really $(x-2)^2 + (y-0)^2 = 5^2$, and **the center must be at $(h, k) = (2, 0)$**. Clearly, **the radius is $r = 5$**.

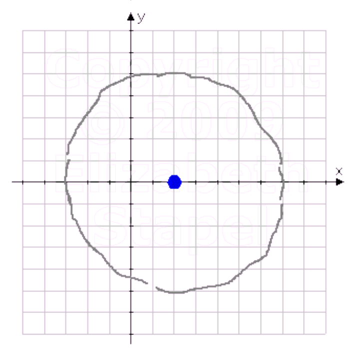
To sketch, I'll first draw the dot for the center:



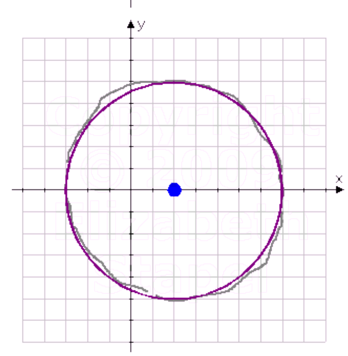
Then I'll rough in markers that are five units away from the center in each of the four "easy" directions:



Then I'll rough in the curvy bits in between these markers, turning the paper as I go:



I'll make whatever corrections look useful, trying to make my circle look properly circular, and draw my final answer as a solid dark line:



It can be very helpful to rotate your paper when you do your drawing, so as to end up with a circle that looks fairly round.

1) **State the radius and center of the circle with equation $16 = (x - 2)^2 + (y - 3)^2$.**

The numerical side, the 16, is the *square* of the radius, so it actually indicates $16 = r^2 = 4^2$, so **the radius is $r = 4$**. Reading from the squared-variable parts, **the center is at $(h, k) = (2, 3)$**.

- **State the radius and center of the circle with equation $25 = x^2 + (y + 3)^2$.**

The numerical side tells me that $r^2 = 25$, so $r = 5$. The x-squared part is really $(x - 0)^2$, so $h = 0$. The temptation is to read off the "3" from the y-squared part and conclude that k is 3, but this is wrong. The center-vertex form has *subtraction* in it, so I need to convert first to that form.

$$y + 3 = y - (-3)$$

So the y-coordinate of the center is actually $k = -3$.

radius $r = 5$, center $(h, k) = (0, -3)$

Warning: It is very easy to forget that sign in the middle of the squared parts. Don't be careless!

In the previous examples, information was extracted from a given equation. You'll also need to be able to work from given information backwards to find an equation.

- **Find an equation for the circle with center $(h, k) = (4, -2)$ and radius $r = 10$.**

I'll just plug the center and radius into the center-radius form:

$$(x - (4))^2 + (y - (-2))^2 = 10^2$$
$$\mathbf{(x - 4)^2 + (y + 2)^2 = 100}$$

Since no particular form of the equation was specified, the above is an acceptable answer. If your book specifies some other format, then you may need to multiply things out:

$$x^2 - 8x + 16 + y^2 + 4y + 4 = 100$$
$$x^2 + y^2 - 8x + 4y + 20 = 100$$
$$x^2 + y^2 - 8x + 4y - 80 = 0$$

Keep in mind that there is no standard meaning to the term "standard form". If your book specifies some other form, *memorize* that form for your tests.

- **Find the center and radius of the circle with equation $x^2 + y^2 + 2x + 8y + 8 = 0$**

  To convert to center-radius form, I'll need to complete the squares.

$$x^2 + y^2 + 2x + 8y + 8 = 0$$
$$x^2 + 2x + [\ ] + y^2 + 8y + [\ ] = -8 + [\ ] + [\ ]$$
$$x^2 + 2x + [\ 1\ ] + y^2 + 8y + [16] = -8 + [\ 1\ ] + [16]$$
$$(x + 1)^2 + (y + 4)^2 = 9$$
$$(x - (-1))^2 + (y - (-4))^2 = 3^2$$

  Then **the center is at $(h, k) = (-1, -4)$ and the radius is $r = 3$.**

1) **Find an equation for the circle centered at (–5, 12) and passing thru the point (–2, 8).**

- The Distance Formula will give me the radius, which is, after all, the distance between the center and any point on the circle.

$$\begin{aligned} d &= \sqrt{(-5-(-2))^2 + (12-8)^2} \\ &= \sqrt{(-5+2)^2 + (4)^2} \\ &= \sqrt{(-3)^2 + 16} = \sqrt{9+16} \\ &= \sqrt{25} = 5 = r \end{aligned}$$

  Then the center-radius form of the equation is:

$$(x - (-5))^2 + (y - (12))^2 = 5^2$$
$$\mathbf{(x + 5)^2 + (y - 12)^2 = 25}$$

- **Find an equation for the circle with a diameter with endpoints at (9, –4) and (–1, 0).**

  The radius will be half the length of the diameter, and the midpoint of the diameter will be the center. The Midpoint Formula gives me:

$$\begin{aligned} (h, k) &= \left(\frac{9+(-1)}{2},\ \frac{-4+0}{2}\right) \\ &= \left(\frac{8}{2},\ \frac{-4}{2}\right) = (4,\ -2) \end{aligned}$$

  I can use either end of the diameter for my point on the circle; the distance between the center and the circle will be the same, regardless. I like smaller numbers, so I'll pick (–1, 0). The Distance Formula gives me:

$$\begin{aligned} d &= \sqrt{(4-(-1))^2 + (-2-0)^2} \\ &= \sqrt{(4+1)^2 + (-2)^2} = \sqrt{(5)^2 + 4} \\ &= \sqrt{25+4} = \sqrt{29} \end{aligned}$$

This distance is the length of the radius r, so $r^2 = 29$. Plugging my results into the center-radius form, I get:

$$(x - (4))^2 + (y - (-2))^2 = 29$$
$$\mathbf{(x - 4)^2 + (y + 2)^2 = 29}$$

- **Find an equation for the circle centered at (1, –5) and tangent to the line 3*x* + 4*y* = 8.**

For one line (or curve) to be "tangent" to another means that the lines just touch; they don't cross. In the context of circles, the tangent is perpendicular to the radius line at that point on the circle. This means that we need to find the line perpendicular to $3x + 4y = 8$ and passing through the point (1, –5), because the radius will lie on that line. Also, the intersection of the radius line and the given line will be a point on the circle, from which we can find the length of the radius.

First, I find the slope of the given line:

$$3x + 4y = 8$$
$$4y = -3x + 8$$
$$y = -(3/4)x + 2$$

From the slope-intercept form of the line, I can see that the slope of the given line is – 3/4, so my radius line has slope $m = 4/3$. Plugging this and the center point into the point-slope equation of a straight line, I get:

$$y - (-5) = (4/3)(x - 1)$$
$$y + 5 = (4/3)(x - 1)$$
$$y = (4/3)x - (19/3)$$

The intersection of the radius line and the tangent line is a point on the circle, so I'll solve the system of equations represented by these two lines:

$$-(3/4)x + 2 = (4/3)x - (19/3)$$
$$-9x + 24 = 16x - 76$$
$$24+76 = 16x + 9x$$
$$100 = 25x$$
$$4 = x$$

Then $y = -1$, and the point (4, –1) is on the circle. The Distance Formula gives me the length of the radius:

$$d = \sqrt{(1-4)^2 + (-5-(-1))^2} = \sqrt{(-3)^2 + (-5+1)^2}$$
$$= \sqrt{9 + (-4)^2} = \sqrt{9+16} = \sqrt{25} = 5 = r$$

So $r = 5$, and my equation is:

$$\mathbf{(x - 1)^2 + (y + 5)^2 = 25}$$

The "interior" of a circle is all the points inside the circle's line. Algebraically, the distance of an interior point from the center is less than the value of the radius. The "exterior" is all points outside the circle's line, having distances from the center greater than the radius.

- **Is the point (–2, 0) in the interior or exterior of the circle with equation**

$$\mathbf{(x-3)^2+(y+5)^2=49?}$$

The center of the circle is $(h, k) = (3, -5)$ and the radius is $r = 7$. To determine on which "side" of the circle this point lies, I need to find its distance from the center.

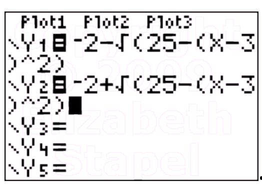
$$\begin{aligned} d &= \sqrt{(3-(-2))^2+(-5-0)^2} = \sqrt{(3+2)^2+(-5)^2} \\ &= \sqrt{5^2+25} = \sqrt{25+25} = \sqrt{50} > \sqrt{49} = 7 = r \end{aligned}$$

Since this distance is more than the radius, then **this point is in the exterior.**

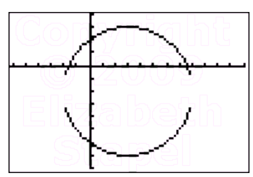
To check your graphs, or to graph an equation to verify that it fits the requirements of an exercise, solve the circle equation for the two half-circle equations. For instance, $(x-3)^2 + (y+2)^2 = 25$ solves as:

$$\begin{aligned} (x-3)^2+(y+2)^2 &= 25 \\ (y+2)^2 &= 25-(x-3)^2 \\ y+2 &= \pm\sqrt{25-(x-3)^2} \\ y &= -2\pm\sqrt{25-(x-3)^2} \end{aligned}$$

The "plus the square root of" part is the top half of the circle; the "minus the square root of" part is the bottom half. Plug these two halves into your calculator as follows:



...and graph:



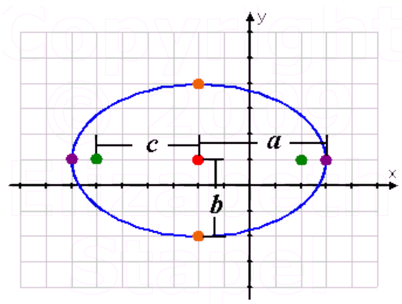
The standard calculator set-up, with the tickmarks on the *x*-axis being wider apart than on the *y*-axis, will make the graph look squashed; the above was done using the "Zoom-Square" setting. Even squashed, though, your graph should suffice to confirm that the circle is centered at or near (3, –2), with a radius at or near 5.

Make sure you memorize the center-radius form of the equation, and practice how to complete the square, how to do quick but nicely round circle graphs, and how to use your graphing calculator to check your answers. Other than involving some messy computations, circles aren't too bad — as conics go.

Ellipses

An ellipse, informally, is an oval or a "squished" circle. In "primitive" geometrical terms, an ellipse is the figure you can draw in the sand by the following process: Push two sticks into the sand. Take a piece of string and form a loop that is big enough to go around the two sticks and still have some slack. Take a third stick, hook it inside the string loop, pull the loop taut by pulling the stick away from the first two sticks, and drag that third stick through the sand at the furthest distance the loop will allow. The resulting shape drawn in the sand is an ellipse.

Each of the two sticks you first pushed into the sand is a "**focus**" of the ellipse; the two together are called "foci". If you draw a line in the sand "through" these two sticks, from one end of the ellipse to the other, this will mark the "major" axis of the ellipse. The points where the major axis touches the ellipse are the "**vertices**" of the ellipse. The point midway between the two sticks is the "**center**" of the ellipse.



If you draw a line through this center, perpendicular to the major axis and from one side of the ellipse to the other, this will mark the "minor" axis. The points where the minor axis touches the ellipse are the "**co-vertices**". A half-axis, from the center out to the ellipse, is called a "semi-major" or a "semi-minor" axis, depending on which axis you're taking half of.

The distance from the center to either focus is the fixed value c. The distance from the center to a vertex is the fixed value a. The values of a and c will vary from one ellipse to another, but they are fixed for any given ellipse. I keep the meaning of these two letters straight by mispronouncing the phrase "foci for c" as "FOH-ciy foh SEE", to remind me that c relates to the focus. Then the other letter (a) is for the other type of point (the vertex).

The length of the semi-major axis is a and the length of the whole major axis is $2a$, and the distance between the foci is $2c$. Okay, so now we've got relationships for a and c, which leads one to wonder, "What happened to b?" The three letters are related by the equation $b^2 = a^2 - c^2$ or, alternatively (depending on your book or instructor), by the equation $b^2 + c^2 = a^2$.

(Proving the relationship requires pages and pages of algebraic computations, so just trust me that the equation is true. It can also be shown — painfully — that b is also the length of the semi-minor axis, so the distance across the ellipse in the "shorter" direction is $2b$. Yes, the Pythagorean Theorem is involved in proving this stuff. Yes, these are the same letters used in the Pythagorean Theorem. No, this is *not* the same as the Pythagorean Theorem. Yes, this is very confusing. Accept it, make sure to memorize the relationship before the next test, and move on.)

For a wider-than-tall ellipse with center at (h, k), having vertices a units to either side of the center and foci c units to either side of the center, the ellipse equation is:

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

For a taller-than-wide ellipse with center at (h, k), having vertices a units above and below the center and foci c units above and below the center, the ellipse equation is:

$$\frac{(y-k)^2}{a^2}+\frac{(x-h)^2}{b^2}=1$$

An ellipse equation, in conics form, is always "=1". Note that, in both equations above, the h always stayed with the x and the k always stayed with the y. The only thing that changed between the two equations was the placement of the a^2 and the b^2. The a^2 always goes with the variable whose axis parallels the wider direction of the ellipse; the b^2 always goes with the variable whose axis parallels the narrower direction. Looking at the equations the other way, the larger denominator always gives you the value of a^2, the smaller denominator always gives you the value of b^2, and the two denominators together allow you to find the value of c^2 and the orientation of the ellipse.

Ellipses are, by their nature, not "perfectly round" in the technical sense that circles are round. The measure of the amount by which an ellipse is "squished" away from being perfectly round is called the ellipse's "eccentricity", and the value of an ellipse's eccentricity is denoted as $e = c/a$. Since the foci are closer to the center than are the vertices, then $c < a$, so the value of e will always be less than 1. If an ellipse's foci are pulled inward toward the center, the ellipse will get progressively closer to being a circle. Continuing that process, if we let $c = 0$ (so the foci are actually at the center), this would correspond to $e = 0$, with the ellipse really being a circle. This tells us that the value of e for a true (non-circle) ellipse will always be more than 0. Putting this together, we see that $0 < e < 1$ for any ellipse.

When scientists refer to something (such as Pluto) as having an "eccentric" orbit, they don't mean that the orbit is "weird"; they mean that it's "far from being circular". In Pluto's case, its orbit actually crosses inside that of Neptune from time to time. The larger the value of e, the more "squished" the ellipse.

A physical property of ellipses is that sound or light rays emanating from one focus will reflect back to the other focus. This property can be used, for instance, in medicine. A patient suffering from, say, gall stones can be placed next to a machine that emits shock waves away from the patient and into an elliptical bowl. The patient is carefully positioned so that the gall stones are at one focus of the ellipse, with a water-filled cushion between the machine and the patient allowing for efficient transmission of the shock waves. The machine emits waves from the other focus of the ellipse; these waves scatter harmlessly from the emitter into the elliptical bowl, bounce back from the bowl, and finally reconcentrate at the other focus (inside the patient). The shock waves reach full power only at the patient's focus, where they smash the stone into small enough pieces that the patient's body will be able to get rid of them on its own. The patient can go home the same day, having required no invasive surgery.

Your first task will usually be to demonstrate that you can extract information about an ellipse from its equation, and also to graph a few ellipses.

- **State the center, vertices, foci and eccentricity of the ellipse with general equation $16x^2 + 25y^2 = 400$, and sketch the ellipse.**

  To be able to read any information from this equation, I'll need to rearrange it to get "=1", so I'll divide through by 400. This gives me:

  $$\frac{16x^2}{400} + \frac{25y^2}{400} = \frac{400}{400}$$
  $$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

  Since $x^2 = (x - 0)^2$ and $y^2 = (y - 0)^2$, the equation above is really:

  $$\frac{(x-0)^2}{5^2} + \frac{(y-0)^2}{4^2} = 1$$

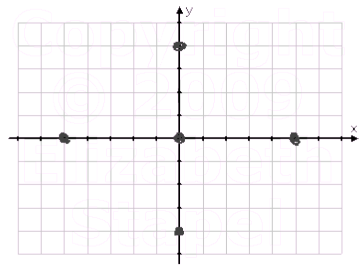
  Then the center is at $(h, k) = (0, 0)$. I know that the a^2 is always the larger denominator (and b^2 is the smaller denominator), and this larger denominator is under the variable that parallels the longer direction of the ellipse. Since 25 is larger than 16, then $a^2 = 25$, $a = 5$, and this ellipse is wider (paralleling the x-axis) than it is tall. The value of a also tells me that the vertices are five units to either side of the center, at $(-5, 0)$ and $(5, 0)$.

  To find the foci, I need to find the value of c. From the equation, I already have a^2 and b^2, so:

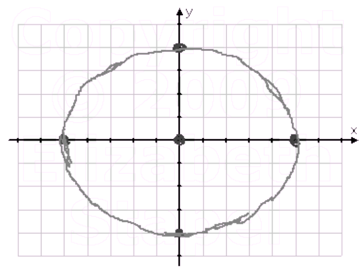
  $$a^2 - c^2 = b^2$$
  $$25 - c^2 = 16$$
  $$9 = c^2$$

  Then the value of c is 3, and the foci are three units to either side of the center, at $(-3, 0)$ and $(3, 0)$. Also, the value of the eccentricity e is $c/a = 3/5$.

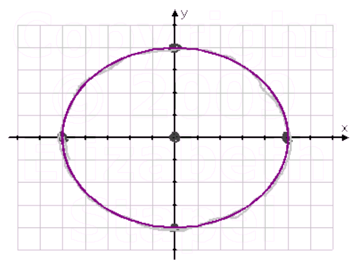
To sketch the ellipse, I first draw the dots for the center and the endpoints of each axis:



Then I rough in a curvy line, rotating my paper as I go and eye-balling my curve for smoothness...



...and then I draw my "answer" as a heavier solid line.



center (0, 0), vertices (–5, 0) and (5, 0), foci (–3, 0) and (3, 0), and eccentricity 3/5

You may find it helpful to do the roughing in with pencil, rotating the paper as you go around, and then draw your final graph in pen, carefully erasing your "rough draft" before you hand in your work. And always make sure your graph is neat and is large enough to be clear.

- **State the center, foci, vertices, and co-vertices of the ellipse with equation $25x^2 + 4y^2 + 100x - 40y + 100 = 0$. Also state the lengths of the two axes.**

  I first have to rearrange this equation into conics form by completing the square and dividing through to get "=1". Once I've done that, I can read off the information I need from the equation.

$$25x^2 + 4y^2 + 100x - 40y = -100$$
$$25x^2 + 100x + 4y^2 - 40y = -100$$
$$25(x^2 + 4x\ ) + 4(y^2 - 10y\ ) = -100 + 25(\ ) + 4(\ )$$
$$25(x^2 + 4x + 4) + 4(y^2 - 10y + 25) = -100 + 25(\ 4\ ) + 4(\ 25\ )$$
$$25(x + 2)^2 + 4(y - 5)^2 = -100 + 100 + 100 = 100$$
$$\frac{25(x+2)^2}{100} + \frac{4(y-5)^2}{100} = \frac{100}{100}$$
$$\frac{(x+2)^2}{4} + \frac{(y-5)^2}{25} = 1$$

The larger demoninator is a^2, and the y part of the equation has the larger denominator, so this ellipse will be taller than wide (to parallel the y-axis). Also, $a^2 = 25$ and $b^2 = 4$, so the equation $b^2 + c^2 = a^2$ gives me $4 + c^2 = 25$, and c^2 must equal 21. The center is clearly at the point $(h, k) = (–2, 5)$. The vertices are $a = 5$ units above and below the center, at (–2, 0) and (–2, 10). The co-vertices are $b = 2$ units to either side of the center, at (–4, 5) and (0, 5). The major axis has length $2a = 10$, and the minor axis has length $2b = 4$. The foci are messy: they're *sqrt*[21] units above and below the center.

center (–2, 5), vertices (–2, 0) and (–2, 10),
co-vertices (–4, 5) and (0, 5), foci $\boldsymbol{(-2, 5-\sqrt{25}}$**and** $\boldsymbol{(-2, 5\sqrt{25}}$**,**
major axis length 10, minor axis length 4

You'll also need to work the other way, finding the equation for an ellipse from a list of its properties.

- **Write an equation for the ellipse having one focus at (0, 3), a vertex at (0, 4), and its center at (0, 0).**

  Since the focus and vertex are above and below each other, rather than side by side, I know that this ellipse must be taller than it is wide. Then a^2 will go with the y part of the equation. Also, since the focus is 3 units above the center, then $c = 3$; since the vertex is 4 units above, then $a = 4$. The equation $b^2 = a^2 – c^2$ gives me $16 – 9 = 7 = b^2$. (Since I wasn't asked for the length of the minor axis or the location of the co-vertices, I don't need the value of b itself.) Then my equation is:

$$\boldsymbol{\frac{y^2}{16} + \frac{x^2}{7} = 1}$$

- **Write an equation for the ellipse with vertices (4, 0) and (–2, 0) & foci (3, 0) & (–1, 0).**

  The center is midway between the two foci, so $(h, k) = (1, 0)$, by the Midpoint Formula. Each focus is 2 units from the center, so $c = 2$. The vertices are 3 units from the center, so $a = 3$. Also, the foci and vertices are to the left and right of each other, so this ellipse is wider than it is tall, and a^2 will go with the x part of the ellipse equation.

  The equation $b^2 = a^2 – c^2$ gives me $9 – 4 = 5 = b^2$, and this is all I need to create my equation:

$$\boldsymbol{\frac{(x-1)^2}{9} + \frac{y^2}{9} = 1}$$

- **Write an equation for the ellipse centered at the origin, having a vertex at (0, –5) and containing the point (–2, 4).**

  Since the vertex is 5 units below the center, then this vertex is taller than it is wide, and the a^2 will go with the y part of the equation. Also, $a = 5$, so $a^2 = 25$. I know that $b^2 = a^2 - c^2$, but I don't know the values of b or c. However, I do have the values of h, k, and a, and also a set of values for x and y, those values being the point they gave me on the ellipse. So I'll set up the equation with everything I've got so far, and solve for b.

$$\frac{y^2}{25} + \frac{x^2}{b^2} = 1$$

$$\frac{(4)^2}{25} + \frac{(-2)^2}{b^2} = 1$$

$$\frac{16}{25} + \frac{4}{b^2} = 1$$

$$16b^2 + 100 = 25b^2$$
$$100 = 9b^2$$
$$100/9 = b^2$$

  Then my equation is:

$$\frac{y^2}{25} + \frac{x^2}{\left(\frac{100}{9}\right)} = 1$$

$$\frac{y^2}{25} + \frac{9x^2}{100} = 1$$

- **Write an equation for the ellipse having foci at (–2, 0) and (2, 0) & eccentricity $e = 3/4$.**

  The center is between the two foci, so $(h, k) = (0, 0)$. Since the foci are 2 units to either side of the center, then $c = 2$, this ellipse is wider than it is tall, and a^2 will go with the x part of the equation. I know that $e = c/a$, so $3/4 = 2/a$. Solving the proportion, I get $a = 8/3$, so $a^2 = 64/9$. The equation $b^2 = a^2 - c^2$ gives me $64/9 - 4 = 64/9 - 36/9 = 28/9 = b^2$.

  Now that I have values for a^2 and b^2, I can create my equation:

$$\frac{9x^2}{64} + \frac{9y^2}{28} = 1$$

- **A "whispering room" is one with an elliptically-arched ceiling. If someone stands at one focus of the ellipse and whispers something to his friend, the dispersed sound waves are reflected by the ceiling and concentrated at the other focus, allowing people across the room to clearly hear what he said. Suppose such gallery has a ceiling reaching twenty feet above the five-foot-high vertical walls at its tallest point (so the cross-section is half an ellipse topping two vertical lines at either end), and suppose the foci of the ellipse are thirty feet apart. What is the height of the ceiling above each "whispering point"?**

  Since the ceiling is half of an ellipse (the top half, specifically), and since the foci will be on a line between the tops of the "straight" parts of the side walls, the foci will be five feet above the floor, which sounds about right for people talking and listening: five feet high is close to face-high on most adults.

  I'll center my ellipse above the origin, so (h, k) = (0, 5). The foci are thirty feet apart, so they're 15 units to either side of the center. In particular, $c = 15$. Since the elliptical part of the room's cross-section is twenty feet high above the center, and since this "shorter" direction is the semi-minor axis, then $b = 20$. The equation $b^2 = a^2 - c^2$ gives me $400 = a^2 - 225$, so $a^2 = 625$. Then the equation for the elliptical ceiling is:

$$\frac{(x-0)^2}{625}+\frac{(y-5)^2}{400}=1$$

  I need to find the height of the ceiling above the foci. I prefer positive numbers, so I'll look at the focus to the right of the center. The height (from the ellipse's central line through its foci, up to the ceiling) will be the y-value of the ellipse when $x = 15$:

$$\frac{(15)^2}{625}+\frac{(y-5)^2}{400}=1$$

$$\frac{225}{625}+\frac{(y-5)^2}{400}=1$$

$$\frac{9}{25}+\frac{(y-5)^2}{400}=1$$

$$144 + (y - 5)^2 = 400$$

$$(y - 5)^2 = 256$$

$$y - 5 = \pm 16$$

$$y = 21$$

  (Since I'm looking for the height above, not the depth below, I ignored the negative solution to the quadratic equation.)

  The ceiling is **21 feet above the floor.**

- **Satellites can be put into elliptical orbits if they need only sometimes to be in high- or low-earth orbit, thus avoiding the need for propulsion and navigation in low-earth orbit and the expense of launching into high-earth orbit. Suppose a satellite is in an elliptical orbit, with *a* = 4420 and *b* = 4416, and with the center of the Earth being at one of the foci of the ellipse. Assuming the Earth has a radius of about 3960 miles, find the lowest and highest altitudes of the satellite above the Earth.**

  The lowest altitude will be at the vertex closer to the Earth; the highest altitude will be at the other vertex. Since I need to measure these altitudes from the focus, I need to find the value of c.

$$b^2 = a^2 - c^2$$
$$c^2 = a^2 - b^2 = 4420^2 - 4416^2 = 35{,}344$$

  Then $c = 188$. If I set the center of my ellipse at the origin and make this a wider-than-tall ellipse, then I can put the Earth's center at the point (188, 0).

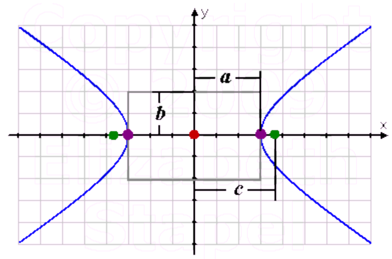
(This means, by the way, that there isn't much difference between the circumference of the Earth and the path of the satellite. The center of the elliptical orbit is actually inside the Earth, and the ellipse, having an eccentricity of $e = 188 / 4420$, or about 0.04, is pretty close to being a circle.)

  The vertex closer to the end of the ellipse containing the Earth's center will be at 4420 units from the ellipse's center, or 4420 – 188 = 4232 units from the center of the Earth. Since the Earth's radius is 3960 units, then the altitude is 4232 – 3960 = 272. The other vertex is 4420 + 188 = 4608 units from the Earth's center, giving me an altitude of 4608 – 3960 = 648 units.

The minimum altitude is 272 miles above the Earth;
the maximum altitude is 648 miles above the Earth.

Hyperbolas

Hyperbolas don't come up much — at least not that I've noticed — in other math classes, but if you're covering conics, you'll need to know their basics. An hyperbola looks sort of like two mirrored parabolas, with the two "halves" being called "branches". Like an ellipse, an hyperbola has two foci and two vertices; unlike an ellipse, the foci in an hyperbola are further from the hyperbola's center than are its vertices:



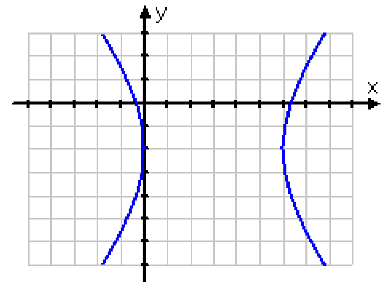
The **hyperbola** is centered on a point (h, k), which is the "**center**" of the hyperbola. The point on each branch closest to the center is that branch's "**vertex**". The vertices are some fixed distance a from the center. The line going from one vertex, through the center, and ending at the other vertex is called the "transverse" axis. The "**foci**" of an hyperbola are "inside" each branch, and each focus is located some fixed distance c from the center. (This means that $a < c$ for hyperbolas.) The values of a and c will vary from one hyperbola to another, but they will be fixed values for any given hyperbola.

For any point on an ellipse, the *sum* of the distances from that point to each of the foci is some fixed value; for any point on an hyperbola, it's the *difference* of the distances from the two foci that is fixed. Looking at the graph above and letting "the point" be one of the vertices, this fixed distance must be (the distance to the further focus) less (the distance to the nearer focus), or $(a + c) - (c - a) = 2a$. This fixed-difference property can used for determining locations: If two beacons are placed in known and fixed positions, the difference in the times at which their signals are received by, say, a ship at sea can tell the crew where they are.

As with ellipses, there is a relationship between a, b, and c, and, as with ellipses, the computations are long and painful. So trust me that, for hyperbolas (where $a < c$), the relationship is $c^2 - a^2 = b^2$ or, which means the same thing, $c^2 = b^2 + a^2$. (Yes, the Pythagorean Theorem is used to prove this relationship. Yes, these are the same letters as are used in the Pythagorean Theorem. No, this is not the same thing as the Pythagorean Theorem. Yes, this is very confusing. Just memorize it, and move on.)

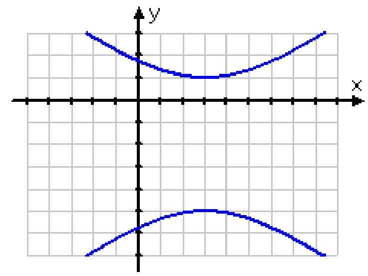
When the transverse axis is horizontal (in other words, when the center, foci, and vertices line up side by side, parallel to the *x*-axis), then the a^2 goes with the *x* part of the hyperbola's equation, and the *y* part is subtracted.

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$



When the transverse axis is vertical (in other words, when the center, foci, and vertices line up above and below each other, parallel to the *y*-axis), then the a^2 goes with the *y* part of the hyperbola's equation, and the *x* part is subtracted.

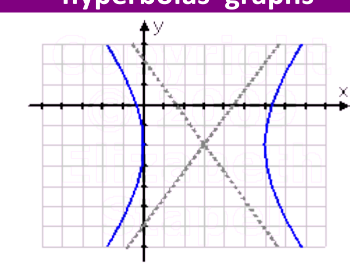
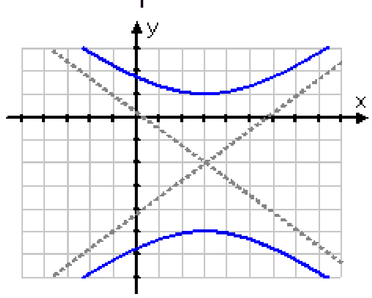
$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$



In "conics" form, an hyperbola's equation is always "=1".

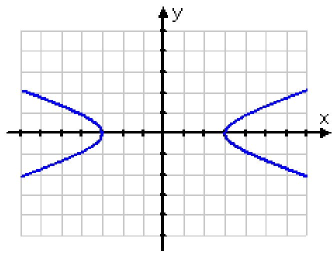
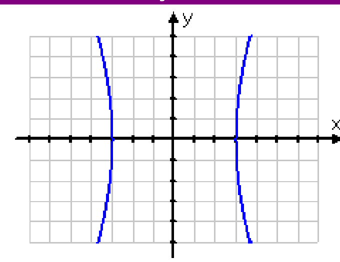
The value of *b* gives the "height" of the "fundamental box" for the hyperbola (**marked in grey** in the first picture above), and 2*b* is the length of the "conjugate" axis. This information doesn't help you graph hyperbolas, though.

For reasons you'll learn in calculus, the graph of an hyperbola gets fairly flat and straight when it gets far away from its center. If you "zoom out" from the graph, it will look very much like an "X", with maybe a little curviness near the middle. These "nearly straight" parts get very close to what are called the "<u>asymptotes</u>" of the hyperbola. For an hyperbola centered at (*h*, *k*) and having fixed values *a* and *b*, the asymptotes are given by the following equations:

hyperbolas' graphs	asymptotes' equations
	$y=\pm\frac{b}{a}(x-h)+k$
	$y=\pm\frac{a}{b}(x-h)+k$

Note that the only difference in the asymptote equations above is in the slopes of the straight lines: If a^2 is the denominator for the x part of the hyperbola's equation, then a is still in the denominator in the slope of the asymptotes' equations; if a^2 goes with the y part of the hyperbola's equation, then a goes in the numerator of the slope in the asymptotes' equations.

Hyperbolas can be fairly "straight" or else pretty "bendy":

hyperbola with an eccentricity of about 1.05	hyperbola with an eccentricity of about 7.6
	

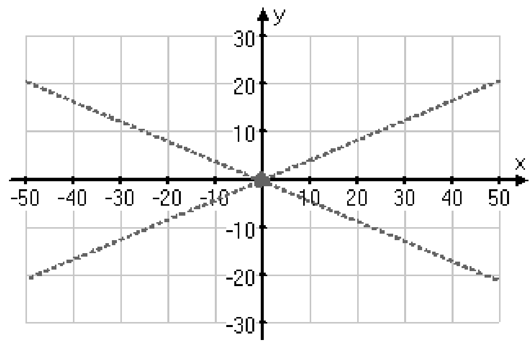
The measure of the amount of curvature is the "eccentricity" e, where $e = c/a$. Since the foci are further from the center of an hyperbola than are the vertices (so $c > a$ for hyperbolas), then $e > 1$. Bigger values of e correspond to the "straighter" types of hyperbolas, while values closer to 1 correspond to hyperbolas whose graphs curve quickly away from their centers.

- **Find the center, vertices, foci, eccentricity, and asymptotes of the hyperbola with the given equation, and sketch:** $\frac{y^2}{25} - \frac{x^2}{144} = 1$

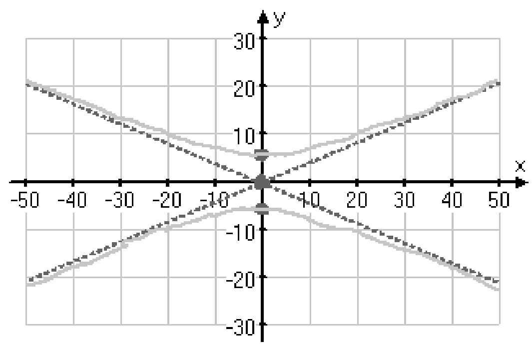
  Since the y part of the equation is added, then the center, foci, and vertices will be above and below the center (on a line paralleling the y-axis), rather than side by side.

  Looking at the denominators, I see that $a^2 = 25$ and $b^2 = 144$, so $a = 5$ and $b = 12$. The equation $c^2 - a^2 = b^2$ tells me that $c^2 = 144 + 25 = 169$, so $c = 13$, and the eccentricity is $e = 13/5$. Since $x^2 = (x - 0)^2$ and $y^2 = (y - 0)^2$, then the center is at $(h, k) = (0, 0)$. The vertices and foci are above and below the center, so the foci are at $(0, -13)$ and $(0, 13)$, and the vertices are at $(0, 5)$ and $(0, -5)$.

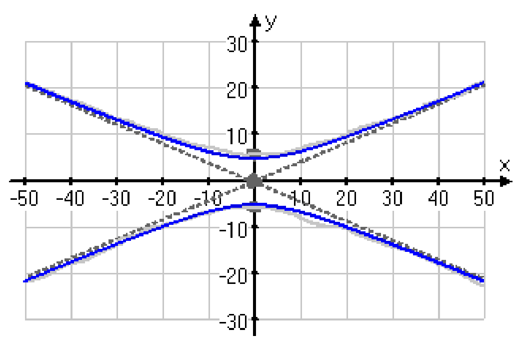
Because the *y* part of the equation is dominant (being added, not subtracted), then the slope of the asymptotes has the *a* on top, so the slopes will be *m* = ± 5/12. To graph, I start with the center, and draw the asymptotes through it, using dashed lines:



Then I draw in the vertices, and rough in the graph, rotating the paper as necessary and "eye-balling" for smoothness:



Then I draw in the final graph as a neat, smooth, heavier line:



And the rest of my answer is:

center (0, 0), vertices (0, –5) and (0, 5), foci (0, –13) and (0, 13), eccentricity $e = \frac{13}{5}$**, and asymptotes** $y = \pm\frac{5}{12}x$

- **Give the center, vertices, foci, and asymptotes for the hyperbola with equation:** $\frac{(x+3)^2}{16} - \frac{(y-2)^2}{9} = 1$

Since the *x* part is added, then $a^2 = 16$ and $b^2 = 9$, so $a = 4$ and $b = 3$. Also, this hyperbola's foci and vertices are to the left and right of the center, on a horizontal line paralleling the *x*-axis.

From the equation, clearly the center is at $(h, k) = (-3, 2)$. Since the vertices are $a = 4$ units to either side, then they are at $(-7, 2)$ and at $(1, 2)$. The equation $c^2 - a^2 = b^2$ gives me $c^2 = 9 + 16 = 25$, so $c = 5$, and the foci, being 5 units to either side of the center, must be at $(-8, 2)$ and $(2, 2)$.

Since the a^2 went with the x part of the equation, then a is in the denominator of the slopes of the asymptotes, giving me $m = \pm 3/4$. Keeping in mind that the asymptotes go through the center of the hyperbola, the asymptes are then given by the straight-line equations $y - 2 = \pm (3/4)(x + 3)$.

center (–3, 2), vertices (–7, 2) and (1, 2), foci (–8, 2) and (2, 2),
and asymptotes $\boldsymbol{y = \pm\frac{3}{4}(x+3)+2}$

- **Find the center, vertices, and asymptotes of the hyperbola with equation $\boldsymbol{4x^2 - 5y^2 + 40x - 30y - 45 = 0}$.**

To find the information I need, I'll first have to convert this equation to "conics" form by completing the square.

$$4x^2 + 40x - 5y^2 - 30y = 45$$
$$4(x^2 + 10x\ ) - 5(y^2 + 6y\ ) = 45 + 4(\ ) - 5(\ )$$
$$4(x^2 + 10x + 25) - 5(y^2 + 6y + 9) = 45 + 4(25) - 5(9)$$
$$4(x + 5)^2 - 5(y + 3)^2 = 45 + 100 - 45$$

$$\frac{4(x+5)^2}{100} - \frac{5(y+3)^2}{100} = \frac{100}{100}$$

$$\frac{(x+5)^2}{25} - \frac{(y+3)^2}{20} = 1$$

Then the center is at $(h, k) = (-5, -3)$. Since the x part of the equation is added, then the center, foci, and vertices lie on a horizontal line paralleling the x-axis; $a^2 = 25$ and $b^2 = 20$, so $a = 5$ and $b = 2$*sqrt*[5]. The equation $a^2 + b^2 = c^2$ gives me $c^2 = 25 + 20 = 45$, so $c =$ *sqrt*[45] = 3*sqrt*[5]. The slopes of the two asymptotes will be $m = \pm$ (2/5)*sqrt*[5]. Then my complete answer is:

center (–5, –3), vertices (–10, –3) and (0, –3),

foci $\boldsymbol{(-5 - 3\sqrt{5}, -3)}$**and** $\boldsymbol{(-5 - 3\sqrt{5}, -3)}$**,**

and asymptotes $\boldsymbol{y = \pm\frac{2\sqrt{5}}{5}(x+5)-3}$

If I had needed to graph this hyperbola, I'd have used a decimal approximation of $\pm$ 0.89442719... for the slope, but would have rounded the value to something reasonable like $m = \pm 0.9$.

- **Find an equation for the hyperbola with center (2, 3), vertex (0, 3), and focus (5, 3).**

The center, focus, and vertex all lie on the horizontal line $y = 3$ (that is, they're side by side on a line paralleling the x-axis), so the branches must be side by side, and the x part of the equation must be added. The a^2 will go with the x part of the equation, and the y part will be subtracted. The vertex is 2 units from the center, so $a = 2$; the focus is 3 units from the center, so $c = 3$. Then $a^2 + b^2 = c^2$ gives me $b^2 = 9 - 4 = 5$. I don't need to bother with the value of b itself, since they only asked me for the equation, which is:

$$\frac{(x-2)^2}{4} - \frac{(y-3)^2}{5} = 1$$

- **Find an equation for the hyperbola with center (0, 0), vertex (0, 5), and asymptotes $y = \pm (5/3)x$.**

The vertex and the center are both on the vertical line $x = 0$ (that is, on the y-axis), so the hyperbola's branches are above and below each other, not side by side. Then the y part of the equation will be added, and will get the a^2 as its denominator. Also, the slopes of the two asymptotes will be of the form $m = \pm a/b$.

The vertex they gave me is 5 units above the center, so $a = 5$ and $a^2 = 25$. The slope of the asymptotes (ignoring the "plus-minus" part) is $a/b = 5/3 = 5/b$, so $b = 3$ and $b^2 = 9$. And this is all I need in order to find my equation:

$$\frac{y^2}{25} - \frac{x^2}{9} = 1$$

- **Find an equation of the hyperbola with x-intercepts at $x = -5$ and $x = 3$, and foci at (–6, 0) and (4, 0).**

The foci are side by side, so this hyperbola's branches are side by side, and the center, foci, and vertices lie on a line paralleling the x-axis. So the y part of the equation will be subtracted and the a^2 will go with the x part of the equation. The center is midway between the foci, so the center must be at $(h, k) = (-1, 0)$. The foci are 5 units to either side of the center, so $c = 5$ and $c^2 = 25$.

The center lies on the x-axis, so the two x-intercepts must then also be the hyperbola's vertices. Since the intercepts are 4 units to either side of the center, then $a = 4$ and $a^2 = 16$. Then $a^2 + b^2 = c^2$ tells me that $b^2 = 25 - 16 = 9$, and my equation is:

$$\frac{(x+1)^2}{16} - \frac{y^2}{9} = 1$$

- **Find an equation for the hyperbola with vertices at (–2, 15) and (–2, –1), and having eccentricity *e* = 17/8**.

  The vertices are above and below each other, so the center, foci, and vertices lie on a vertical line paralleling the *y*-axis. Then the a^2 will go with the *y* part of the hyperbola equation, and the *x* part will be subtracted.

  The center is midway between the two vertices, so (*h*, *k*) = (–2, 7). The vertices are 8 units above and below the center, so $a = 8$ and $a^2 = 64$. The eccentricity is $e = c/a = 17/8 = c/8$, so $c = 17$ and $c^2 = 289$. The equation $a^2 + b^2 = c^2$ tells me that $b^2 = 289 – 64 = 225$. Then my equation is:

$$\frac{(y-7)^2}{64} - \frac{(x+2)^2}{225} = 1$$

Algebra 2 Semester 2 Self-Test

Use the zero-factor property to solve the following equations.

(1) $x^2 = 10x - 24$

(a) $\{4,6\}$

(b) $\{-4,-6\}$

(c) $\{-24,-1\}$

(d) $\{1,24\}$

(2) $x^2 = 49$

(a) $\{7,-7\}$

(b) $\{24,5\}$

(c) $\{8,-8\}$

(d) $\{7\}$

Use the quadratic equation to solve the following equations.

(3) $2n^2 = -10n - 7$

(a) $\{\frac{-10+\sqrt{11}}{2}, \frac{-10-\sqrt{11}}{2}\}$

(b) $\{\frac{-10+\sqrt{11}}{2}, \frac{-10-\sqrt{11}}{2}\}$

(c) $\{\frac{-5+\sqrt{11}}{2}, \frac{-5-\sqrt{11}}{2}\}$

(d) $\{\frac{-5+\sqrt{11}}{4}, \frac{-5-\sqrt{11}}{4}\}$

(4) $x^2 + x + 4 = 0$

(a) $\{\frac{1+i\sqrt{15}}{2}, \frac{1-i\sqrt{15}}{2}\}$

(b) $\{\frac{-1+i\sqrt{15}}{2}, \frac{-1-i\sqrt{15}}{2}\}$

(c) $\{\frac{1+\sqrt{15}}{2}, \frac{1-\sqrt{15}}{2}\}$

(d) $\{\frac{-1+\sqrt{15}}{2}, \frac{-1-\sqrt{15}}{2}\}$

Solve the equation by completing the square.

(5) $z^2 + 16z + 44 = 0$

(a) $\{8 + 2\sqrt{5}\}$

(b) $\{8 + 2\sqrt{11}, 8 - 2\sqrt{11}\}$

(c) $\{-16 + 2\sqrt{11}\}$

(d) $\{-8 + 2\sqrt{5}, -8 - 2\sqrt{5}\}$

Use the discriminant to determine whether the equation has two rational solutions, one rational solution, two irrational solutions, or two nonreal complex solutions. Do not actually solve.

(6) $s^2 + 3s - 4 = 0$
- (a) Two nonreal complex solutions
- (b) One rational solution
- (c) Two irrational solutions
- (d) Two rational solutions

(7) $t^2 + 8t + 16 = 0$
- (a) Two nonreal complex solutions
- (b) One rational solution
- (c) Two irrational solutions
- (d) Two rational solutions

(8) $4y^2 = 6y - 7$
- (a) Two nonreal complex solutions
- (b) One rational solution
- (c) Two irrational solutions
- (d) Two rational solutions

(9) Write the logarithmic expression as a single logarithm: $2\ log_b q + 3\ log_b y$
- (a) $(2 + 3)log_b(q + y)$
- (b) $log_b(q^2y^3)$
- (c) $log_b(q^2{+}y^3)$
- (d) $log_b(qy^{2+3})$

(10) Expand the logarithmic expression: $log_4 \frac{3}{x}$
- (a) $\frac{log_4 3}{log_4 x}$
- (b) $-xlog_4 3$
- (c) $log_4 x - log_4 3$
- (d) $log_4 3 - log_4 x$

Evaluate the following expressions

(11) $log\ [\frac{1}{1000}]$
- (a) 3
- (b) -3
- (c) $-\frac{1}{3}$
- (d) $\frac{1}{1000}$

(12) $\ln e$
- (a) 1
- (b) -1
- (c) 0
- **(d)** e

Solve the exponential equation. Express the solution set in terms of natural logarithms.

(13) $e^{5x} = 2$
- (a) $\{\frac{\ln 2}{5}\}$
- (b) $\{5 \ln 2\}$
- (c) $\{\frac{\ln 5}{2}\}$
- (d) $\{\frac{2}{5}e\}$

(14) Identify the conic section represented by the equation $x^2 + 6x + 4y^2 = 9$
- (a) Parabola
- (b) Circle
- (c) Ellipse
- (d) Hyperbola

(15) Identify the conic section represented by the equation $x^2 + 6x + 4y^2 + 12y = 9$
- (a) Parabola
- (b) Circle
- (c) Ellipse
- (d) Hyperbola

(16) Identify the conic section represented by the equation $x^2 + 6x + y^2 - 18y = 9$
- (a) Parabola
- (b) Circle
- (c) Ellipse
- (d) Hyperbola

(17) Identify the conic section represented by the equation $x^2 + 6x - 4y = 9$
- (a) Parabola
- (b) Circle
- (c) Ellipse
- (d) Hyperbola

(18) What is the center of the circle $x^2 + y^2 + 4x - 8y + 10 = 0$?
- (a) $(-2,4)$
- (b) $(2,4)$
- (c) $(4,2)$
- (d) $(4,-2)$

(19) Which are the foci of the hyperbola $16x^2 - 9y^2 = 144$?
- (a) $(\pm 5,0)$
- (b) $(0,\pm 5)$
- (c) $(\pm 3,0)$
- (d) $(0,\pm 3)$